The Open University

M337

Complex analysis

Book C

Geometric methods in complex analysis

This publication forms part of an Open University module. Details of this and other Open University modules can be obtained from Student Recruitment, The Open University, PO Box 197, Milton Keynes MK7 6BJ, United Kingdom (tel. +44 (0)300 303 5303; email general-enquiries@open.ac.uk).

Alternatively, you may visit the Open University website at www.open.ac.uk where you can learn more about the wide range of modules and packs offered at all levels by The Open University.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published 2018.

Edited, designed and typeset by The Open University, using LaTeX.

Printed in the United Kingdom by Henry Ling Limited, at the Dorset Press, Dorchester, DT1 1HD.

ISBN 978 1 4730 2364 2

2.1

Contents

Unit C1

Residues

Introduction

In Book B we obtained many theoretical results about analytic functions and we pointed out that these can often be used to evaluate integrals. For example, if f is a function that is analytic on a simply connected region $\mathcal{R}$, Γ is a simple-closed contour in $\mathcal{R}$, and α is any point inside Γ, then

$$\int_\Gamma f(z)\,dz = 0,$$

by Cauchy's Theorem (Theorem 1.2 of Unit B2),

$$\int_\Gamma \frac{f(z)}{z-\alpha}\,dz = 2\pi i f(\alpha),$$

by Cauchy's Integral Formula (Theorem 2.1 of Unit B2), and

$$\int_\Gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,dz = 2\pi i \frac{f^{(n)}(\alpha)}{n!}, \quad \text{for } n = 1, 2, \dots,$$

by Cauchy's nth Derivative Formula (Theorem 3.2 of Unit B2). Also, if f is analytic on the punctured disc $D = \{z : 0 < |z-\alpha| < r\}$ (which is not a simply connected region), and C is a circle in D centred at α, then

$$\int_C \frac{f(z)}{(z-\alpha)^{n+1}}\,dz = 2\pi i a_n,$$

where a_n is the coefficient of $(z-\alpha)^n$ in the Laurent series about α for f. In particular, when $n = -1$, we have

$$\int_C f(z)\,dz = 2\pi i a_{-1} \tag{0.1}$$

(equation (4.2) of Unit B4). The complex number a_{-1} is called the *residue* of f at α. In this unit we will use residues to evaluate more general complex integrals, and hence solve a number of problems from real analysis.

We start, in Section 1, by introducing a number of useful techniques for evaluating residues.

In Section 2 we state and prove a key result called the Residue Theorem, which gives a formula for evaluating integrals that is more general than equation (0.1). We then apply the Residue Theorem to the evaluation of a wide range of integrals. In particular, we show how it can be used to evaluate certain real trigonometric integrals involving $\cos t$ and $\sin t$, such as

$$\int_0^{2\pi} \frac{1}{5+4\sin t}\,dt \quad \text{and} \quad \int_0^{2\pi} \frac{1}{16\cos^2 t + 9}\,dt.$$

Section 3 is about using the Residue Theorem to evaluate certain real improper integrals of the forms

$$\int_{-\infty}^{\infty} f(t)\,dt, \quad \int_{-\infty}^{\infty} f(t)\cos kt\,dt \quad \text{and} \quad \int_{-\infty}^{\infty} f(t)\sin kt\,dt,$$

where f is a rational function (with real coefficients) subject to certain restrictions, and k is a real number.

In Section 4 we see how residues can also be used to determine the sums of certain real infinite series. For example, we show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Finally, Section 5 is about *analytic continuation*, the process of extending the domain of a given analytic function while preserving the function's analyticity. There we will see more examples of real improper integrals evaluated using the Residue Theorem.

After solving so many problems involving real functions by using methods of complex analysis, you will be able to appreciate the following observation by the French mathematician and twice prime minister of France, Paul Painlevé (1863–1933):

> [B]etween two truths of the real domain, the easiest and shortest path quite often passes through the complex domain.
>
> (Painlevé, 1967, p. 2)

Unit guide

This is one of the longest and most challenging of all the M337 units, but it is also one of the most rewarding. It leads you to results of fundamental importance in complex analysis, most notably the Residue Theorem, which is widely applied in mathematics, engineering and physics.

Section 1 describes various methods for calculating residues. You should familiarise yourself with these as they will be needed throughout the rest of the unit.

As you study Section 2, you should make sure that you understand the statement and use of the Residue Theorem. If you are short of time, then you may prefer to omit the proof of the Residue Theorem on a first reading, and instead proceed to Section 3.

Sections 3 and 4 demonstrate some striking applications of the Residue Theorem. The techniques used are central to complex analysis.

Section 5 is about analytic continuation, a subject that we will return to in Unit C2.

1 Calculating residues

After working through this section, you should be able to:

- calculate residues using Laurent series
- calculate residues at simple poles and removable singularities using the g/h Rule and the Cover-up Rule
- calculate residues at higher-order poles.

1.1 Using Laurent series

Let us begin by revisiting some important definitions from Unit B4. We recall that any function f that is analytic on an open punctured disc $\{z : 0 < |z - \alpha| < r\}$ centred at a point α can be represented by a series

$$f(z) = \cdots + \frac{a_{-2}}{(z-\alpha)^2} + \frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots,$$

for $0 < |z - \alpha| < r$. This series is called the *Laurent series about α for f*, and it is unique: it is the only extended power series about α that represents f on the punctured disc $\{z : 0 < |z - \alpha| < r\}$. The coefficient a_{-1} plays a special role in complex analysis. It is called the *residue* of f at α, and is denoted by $\operatorname{Res}(f, \alpha)$.

Here we look at methods for determining $\operatorname{Res}(f, \alpha)$. One method is to simply inspect the series. The following example will show you some of the techniques involved.

Example 1.1

Find the residue of each of the following functions at the point 0.

(a) $f(z) = \dfrac{1+z^2}{z}$ (b) $f(z) = \dfrac{\sin z}{z^4}$

Solution

(a) The Laurent series about 0 for f is

$$\frac{1+z^2}{z} = \cdots + \frac{0}{z^3} + \frac{0}{z^2} + \frac{1}{z} + 0 + z + 0z^2 + 0z^3 + \cdots,$$

so $\operatorname{Res}(f, 0) = 1$, the coefficient of z^{-1}.

(b) The Taylor series about 0 for the sine function is

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots.$$

Dividing by z^4, we obtain the Laurent series about 0 for f,

$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!\,z} + \frac{z}{5!} - \cdots,$$

so $\operatorname{Res}(f, 0) = -1/6$, the coefficient of z^{-1}.

Exercise 1.1

Find the residue of each of the following functions at the point 0.

(a) $f(z) = \dfrac{1}{z^2} - 3$ (b) $f(z) = \dfrac{1}{z-1}$ (c) $f(z) = \dfrac{\cos z}{z^3}$

(d) $f(z) = z^2 \sin(1/z)$

It sometimes helps to rearrange the function into a more useful form when finding residues.

Example 1.2

Find the residue of the function

$$f(z) = \frac{1}{z(z-1)^2}$$

at each of the following points α.

(a) $\alpha = 0$ (b) $\alpha = 1$

Solution

(a) We need to find the Laurent series about 0 for f. Using the binomial series expansion of $(1-z)^{-2}$, which is valid for $|z| < 1$, we see that if $0 < |z| < 1$, then

$$\begin{aligned}\frac{1}{z(z-1)^2} &= \frac{1}{z}(1-z)^{-2}\\ &= \frac{1}{z}\left(1 + \frac{(-2)}{1!}(-z) + \frac{(-2)\times(-3)}{2!}(-z)^2 + \cdots\right)\\ &= \frac{1}{z}(1 + 2z + 3z^2 + \cdots)\\ &= \frac{1}{z} + 2 + 3z + \cdots.\end{aligned}$$

Hence $\operatorname{Res}(f, 0) = 1$, the coefficient of z^{-1}.

(b) We need to expand the given function in powers of $z - 1$. To do this, we substitute $w = z - 1$, so $z = 1 + w$. We obtain, for $0 < |w| < 1$ (or, equivalently, $0 < |z - 1| < 1$),

$$\begin{aligned}\frac{1}{z(z-1)^2} &= \frac{1}{(1+w)w^2}\\ &= \frac{1}{w^2}(1+w)^{-1}\\ &= \frac{1}{w^2}(1 - w + w^2 - w^3 + \cdots)\\ &= \frac{1}{w^2} - \frac{1}{w} + 1 - w + \cdots\\ &= \frac{1}{(z-1)^2} - \frac{1}{(z-1)} + 1 - (z-1) + \cdots.\end{aligned}$$

Hence $\operatorname{Res}(f, 1) = -1$, the coefficient of $(z-1)^{-1}$.

Since the residue of f at the point α is the coefficient of $(z-\alpha)^{-1}$ in the Laurent series about α for f, it follows on writing $w = z - \alpha$ that this residue is simply the coefficient of w^{-1} in the Laurent series about 0 in powers of w. You may wish to use this observation in the following exercise.

Exercise 1.2

Find the residue of each of the following functions at the given point α.

(a) $f(z) = \dfrac{1}{z^2+1}, \quad \alpha = i$ (b) $f(z) = \dfrac{ze^{iz}}{(z-\pi)^2}, \quad \alpha = \pi$

The next theorem can sometimes save effort in finding residues. It uses the concepts of odd and even functions, which we encountered in Subsection 3.1 of Unit B3. There we saw that if A is a set for which $z \in A$ if and only if $-z \in A$, and f is a function with domain A, then:

f is an odd function if $f(-z) = -f(z)$, for all $z \in A$,

f is an even function if $f(-z) = f(z)$, for all $z \in A$.

For example, the sine function is an odd function and the cosine function is an even function.

Theorem 1.1

Let f be a function that has singularities at the points α and $-\alpha$.

(a) If f is an odd function, then $\operatorname{Res}(f, -\alpha) = \operatorname{Res}(f, \alpha)$.

(b) If f is an even function, then $\operatorname{Res}(f, -\alpha) = -\operatorname{Res}(f, \alpha)$.

Proof Since f has a singularity at α, it has a Laurent series

$$f(z) = \cdots + \frac{a_{-2}}{(z-\alpha)^2} + \frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha) + \cdots$$

about α, which converges on some punctured disc $\{z : 0 < |z-\alpha| < r\}$. If f is an odd or even function, then $z \in A$ if and only if $-z \in A$ (where A is the domain of f), in which case we can substitute $-z$ for z in this series to obtain

$$f(-z) = \cdots + \frac{a_{-2}}{(-z-\alpha)^2} + \frac{a_{-1}}{-z-\alpha} + a_0 + a_1(-z-\alpha) + \cdots.$$

This series converges for $0 < |-z-\alpha| < r$, that is, for $0 < |z+\alpha| < r$. We can simplify the right-hand side to give

$$f(-z) = \cdots + \frac{a_{-2}}{(z+\alpha)^2} - \frac{a_{-1}}{z+\alpha} + a_0 - a_1(z+\alpha) + \cdots. \tag{1.1}$$

Let us now consider the cases when f is an odd and even function in turn.

(a) If f is odd, then $f(-z) = -f(z)$, so we see from equation (1.1) that

$$f(z) = -f(-z) = \cdots - \frac{a_{-2}}{(z+\alpha)^2} + \frac{a_{-1}}{z+\alpha} - a_0 + a_1(z+\alpha) - \cdots,$$

for $0 < |z+\alpha| < r$. Hence $\operatorname{Res}(f, -\alpha) = a_{-1} = \operatorname{Res}(f, \alpha)$.

(b) If f is even, then $f(-z) = f(z)$, so we see from equation (1.1) that

$$f(z) = f(-z) = \cdots + \frac{a_{-2}}{(z+\alpha)^2} - \frac{a_{-1}}{z+\alpha} + a_0 - a_1(z+\alpha) + \cdots,$$

for $0 < |z+\alpha| < r$. Hence $\operatorname{Res}(f, -\alpha) = -a_{-1} = -\operatorname{Res}(f, \alpha)$. ■

To see how this theorem can be applied, consider Exercise 1.2(a) in which you should have found the residue of the *even* function $f(z) = 1/(z^2+1)$ at i to be $-i/2$. Theorem 1.1(b) tells us that

$$\operatorname{Res}(f,-i) = -\operatorname{Res}(f,i) = i/2.$$

Look out for opportunities to apply this theorem in exercises later on!

1.2 Methods for simple poles and removable singularities

Suppose that we wish to calculate the residue at the point $\frac{1}{3}$ of the function

$$f(z) = \frac{1}{z^2(1-z)(1-2z)(1-3z)}.$$

Using the methods of the previous subsection, we would have the difficult task of calculating the coefficient of $\left(z-\frac{1}{3}\right)^{-1}$ in the Laurent series about $\frac{1}{3}$ for f. Fortunately, there are some useful rules for determining residues at simple poles and removable singularities. We turn our attention to these rules in this subsection (and revisit the function f above in Exercise 1.3).

Let f be a function that is analytic on the punctured disc

$$\{z : 0 < |z-\alpha| < r\}$$

and has a simple pole at α. Then f can be represented by a Laurent series about α of the form

$$f(z) = \frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha) + \cdots, \quad \text{for } 0 < |z-\alpha| < r,$$

where $a_{-1} \neq 0$. If we multiply this equation by $z-\alpha$, then we obtain

$$(z-\alpha)f(z) = a_{-1} + a_0(z-\alpha) + a_1(z-\alpha)^2 + \cdots.$$

The power series on the right defines a function that is analytic, and hence continuous, at the point α. It follows that

$$\lim_{z\to\alpha}(z-\alpha)f(z) = a_{-1} = \operatorname{Res}(f,\alpha). \tag{1.2}$$

Note that a similar conclusion holds when the function f has a removable singularity at the point α. The only difference is that $a_{-1} = 0$, and we deduce that

$$\lim_{z\to\alpha}(z-\alpha)f(z) = \operatorname{Res}(f,\alpha) = 0.$$

These observations about simple poles and removable singularities suggest the following result.

Theorem 1.2

Let f be a function that has a singularity at the point α, and suppose that the limit $\lim_{z\to\alpha}(z-\alpha)f(z)$ exists. Then

$$\operatorname{Res}(f,\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z).$$

Furthermore, f has a simple pole at α if the limit is non-zero, and it has a removable singularity at α if the limit is 0.

The statement that the limit $\lim_{z\to\alpha}(z-\alpha)f(z)$ exists means that $(z-\alpha)f(z)$ tends to some complex number as z tends to α.

Proof Let $w = \lim_{z\to\alpha}(z-\alpha)f(z)$.

If $w \neq 0$, then f has a simple pole at α (by Theorem 3.2 of Unit B4), so $\operatorname{Res}(f,\alpha) = w$, by equation (1.2).

If $w = 0$, then f has a removable singularity at α (by Theorem 3.1 of Unit B4), so $\operatorname{Res}(f,\alpha) = w = 0$. ■

We use this result in the following example.

Example 1.3

Find the residue of each of the following functions at the given point α.

(a) $f(z) = \dfrac{1}{z(z-1)^2}$, $\quad \alpha = 0$ (b) $f(z) = \dfrac{1}{z^2+1}$, $\quad \alpha = i$

Solution

(a) Since

$$\lim_{z\to 0}\left((z-0)\times\frac{1}{z(z-1)^2}\right) = \lim_{z\to 0}\frac{1}{(z-1)^2} = 1,$$

we deduce that $\operatorname{Res}(f,0) = 1$.

(b) Since

$$\lim_{z\to i}\left((z-i)\times\frac{1}{z^2+1}\right) = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i},$$

we deduce that $\operatorname{Res}(f,i) = -i/2$.

Compare these solutions with those of Example 1.2(a) and Exercise 1.2(a).

Exercise 1.3

Find the residue of each of the following functions at the given point α.

(a) $f(z) = \dfrac{1}{z^2+4}, \quad \alpha = 2i$

(b) $f(z) = \dfrac{1}{z^2(1-z)(1-2z)(1-3z)}, \quad \alpha = \frac{1}{3}$

Here is a useful corollary to Theorem 1.2.

Corollary g/h Rule

Let $f(z) = g(z)/h(z)$, where g and h are functions that are analytic at the point α, and $h(\alpha) = 0$ and $h'(\alpha) \neq 0$. Then

$$\operatorname{Res}(f, \alpha) = g(\alpha)/h'(\alpha).$$

Proof Since $h(\alpha) = 0$ and $h'(\alpha) \neq 0$, we have

$$\begin{aligned}
\lim_{z\to\alpha}(z-\alpha)\frac{g(z)}{h(z)} &= \lim_{z\to\alpha}\left(g(z) \times \frac{z-\alpha}{h(z)-h(\alpha)}\right) \\
&= \left(\lim_{z\to\alpha} g(z)\right) \Big/ \left(\lim_{z\to\alpha}\frac{h(z)-h(\alpha)}{z-\alpha}\right) \\
&= g(\alpha)/h'(\alpha).
\end{aligned}$$

Therefore $\lim\limits_{z\to\alpha}(z-\alpha)f(z)$ exists and is equal to $g(\alpha)/h'(\alpha)$. It then follows from Theorem 1.2 that $\operatorname{Res}(f, \alpha) = g(\alpha)/h'(\alpha)$. ■

Example 1.4

Find the residue of the function

$$f(z) = \frac{z^2}{z^4-1}$$

at the point i.

Solution

Let $g(z) = z^2$ and $h(z) = z^4 - 1$.

Then g and h are analytic at i. Also, $h(i) = i^4 - 1 = 0$, and $h'(i) = 4i^3$, which is non-zero. Thus the g/h Rule applies, and we have

$$\operatorname{Res}(f, i) = g(i)/h'(i) = i^2/(4i^3) = 1/(4i) = -i/4.$$

Exercise 1.4

Find the residue of each of the following functions at the given points α.

(a) $f(z) = \dfrac{1}{2z^2 + 5iz - 2}, \quad \alpha = -\frac{1}{2}i$

(b) $f(z) = \dfrac{z+9}{(z^2+1)(z^2+9)}, \quad \alpha = 3i$

(c) $f(z) = \dfrac{z^3}{z^4+1}, \quad \alpha = e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}$

The g/h Rule is particularly useful in examples where the function h is a trigonometric or exponential function. The following examples are instances of this.

Example 1.5

Find the residue of each of the following functions at the point n, where n is any integer.

(a) $f(z) = \pi \cot \pi z$ (b) $f(z) = \pi \operatorname{cosec} \pi z$

Solution

(a) Let $g(z) = \pi \cos \pi z$ and $h(z) = \sin \pi z$.

Then $f(z) = g(z)/h(z)$, and g and h are analytic at n. Also,

$$h(n) = \sin \pi n = 0,$$

and

$$h'(n) = \pi \cos \pi n,$$

which is non-zero. Thus the g/h Rule applies, and we obtain

$$\operatorname{Res}(f, n) = \frac{g(n)}{h'(n)} = \frac{\pi \cos \pi n}{\pi \cos \pi n} = 1.$$

(b) Let $g(z) = \pi$ and $h(z) = \sin \pi z$.

Then $f(z) = g(z)/h(z)$, and g and h are analytic at n. Also, $h(n) = 0$ and $h'(n) \neq 0$, as in part (a). Thus the g/h Rule applies, and we obtain

$$\operatorname{Res}(f, n) = \frac{g(n)}{h'(n)} = \frac{\pi}{\pi \cos \pi n} = \frac{1}{(-1)^n} = (-1)^n.$$

Exercise 1.5

Find the residue of each of the following functions at the given points α.

(a) $f(z) = \dfrac{\pi \operatorname{cosec} \pi z}{4z^2 - 1}, \quad \alpha = \frac{1}{2}, -\frac{1}{2}$

(b) $f(z) = \dfrac{\pi \cot \pi z}{4z^2 - 1}, \quad \alpha = \frac{1}{2}, -\frac{1}{2}$

(c) $f(z) = \dfrac{\pi \operatorname{cosec} \pi z}{4z^2 + 1}, \quad \alpha = \frac{1}{2}i, -\frac{1}{2}i$

(d) $f(z) = \dfrac{\pi \cot \pi z}{4z^2 + 1}, \quad \alpha = \frac{1}{2}i, -\frac{1}{2}i$

The residues calculated in Example 1.5 and Exercise 1.5 will be needed in Section 4.

The following special case of the g/h Rule is sufficiently useful to deserve its own name.

Corollary Cover-up Rule

Let $f(z) = \dfrac{g(z)}{z - \alpha}$, where g is a function that is analytic at α. Then

$$\operatorname{Res}(f, \alpha) = g(\alpha).$$

Proof Let $h(z) = z - \alpha$. Then $h(\alpha) = 0$ and $h'(\alpha) = 1$, so

$$\operatorname{Res}(f, \alpha) = g(\alpha)/h'(\alpha) = g(\alpha),$$

by the g/h Rule. ■

The reason for the name 'Cover-up Rule' is that we can find the residue of a function at the simple pole or removable singularity α by covering up the factor $z - \alpha$ in the denominator and evaluating what remains at the point α. As with Theorem 1.2 and the g/h Rule, this method will help you to find residues only at simple poles and removable singularities. Furthermore, you should take heed of the following warning.

When applying the Cover-up Rule, make sure that you cover up only a factor of the form $z - \alpha$.

The following example demonstrates how to apply the Cover-up Rule, and it shows how to deal with factors of the form $kz - \alpha$, where $k \neq 1$.

Example 1.6

Find the residue of the function

$$f(z) = \frac{3z}{(z-i)(2z-i)}$$

at each of the points i and $\frac{1}{2}i$.

Solution

The function f has simple poles at the points i and $\frac{1}{2}i$. If we cover up the factor $z - i$ and evaluate what remains at i, then we obtain

$$\begin{aligned}\operatorname{Res}(f, i) &= \frac{3i}{\boxed{(z-i)}\,(2i-i)} \\ &= \frac{3i}{i} = 3,\end{aligned}$$

by the Cover-up Rule.

For the residue at $\frac{1}{2}i$, we cannot similarly cover up the factor $2z - i$, as this is not of the required form $z - \alpha$. We therefore write

$$f(z) = \frac{3z}{2(z-i)\left(z - \frac{1}{2}i\right)}.$$

If we now cover up the factor $z - \frac{1}{2}i$ and evaluate what remains at $\frac{1}{2}i$, then we obtain

$$\begin{aligned}\operatorname{Res}\left(f, \tfrac{1}{2}i\right) &= \frac{3 \times \left(\frac{1}{2}i\right)}{2\left(\frac{1}{2}i - i\right)\boxed{\left(z - \frac{1}{2}i\right)}} \\ &= \frac{\frac{3}{2}i}{2\left(-\frac{1}{2}i\right)} = -\frac{3}{2},\end{aligned}$$

by the Cover-up Rule.

Exercise 1.6

Use the Cover-up Rule to find the residue of each of the following functions at the given point α.

(a) $f(z) = \dfrac{z+2}{z^3(z+4)}, \quad \alpha = -4$

(b) $f(z) = \dfrac{\cos z}{ze^z}, \quad \alpha = 0$

(c) $f(z) = \dfrac{1}{z^2(1-z)(1-2z)(1-3z)}, \quad \alpha = \frac{1}{3}$

(d) $f(z) = \dfrac{\sin z}{ze^z}, \quad \alpha = 0$

1.3 Methods for higher-order poles

Generally speaking, the calculation of residues at poles of order greater than one is an unpleasant chore, since the useful rules of Subsection 1.2 (the g/h Rule and the Cover-up Rule) do not apply. There are essentially two methods for dealing with higher-order poles: use the Laurent series directly, or use a higher-order analogue of Theorem 1.2. Which method is more appropriate depends on the nature of the function in question.

Using the Laurent series

You have already seen how Laurent series can be used to find residues at poles of order greater than one. For example, in Examples 1.1(b) and 1.2(b) we used Laurent series to calculate the residues of

$$f(z) = \frac{\sin z}{z^4} \text{ at } 0, \quad \text{and} \quad f(z) = \frac{1}{z(z-1)^2} \text{ at } 1.$$

Here are two more such residues for you to calculate.

Exercise 1.7

Use the appropriate Laurent series to find the residue of each of the following functions at the given point α.

(a) $f(z) = \dfrac{z+2}{z^3(z+4)}, \quad \alpha = 0$ (b) $f(z) = \dfrac{1+e^{2z}}{(z-1)^4}, \quad \alpha = 1$

The higher-order formula

If f is a function with a pole of order two at the point α, then its Laurent series about α can be written in the form

$$f(z) = \frac{a_{-2}}{(z-\alpha)^2} + \frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha) + \cdots, \quad \text{where } a_{-2} \neq 0.$$

We wish to isolate the term a_{-1}. It is tempting just to multiply through by $z-\alpha$ and then let z tend to α, but this would cause problems with the first term. Instead, we multiply through by $(z-\alpha)^2$, giving

$$(z-\alpha)^2 f(z) = a_{-2} + a_{-1}(z-\alpha) + a_0(z-\alpha)^2 + a_1(z-\alpha)^3 + \cdots.$$

If we now differentiate each side of this equation, then we obtain

$$\frac{d}{dz}\big((z-\alpha)^2 f(z)\big) = a_{-1} + 2a_0(z-\alpha) + 3a_1(z-\alpha)^2 + \cdots.$$

Taking the limit as z tends to α, we deduce that

$$a_{-1} = \lim_{z\to\alpha}\left(\frac{d}{dz}\big((z-\alpha)^2 f(z)\big)\right).$$

Using a similar method, we can prove the corresponding formula for a pole of order k.

Theorem 1.3

Let f be a function that has a pole of order k at the point α. Then

$$\operatorname{Res}(f,\alpha) = \frac{1}{(k-1)!}\lim_{z\to\alpha}\left(\frac{d^{k-1}}{dz^{k-1}}\Big((z-\alpha)^k f(z)\Big)\right).$$

You are asked to prove Theorem 1.3 in Exercise 1.9 below.

Example 1.7

Find the residue of the function

$$f(z) = \frac{1+e^{2z}}{(z-1)^4}$$

at the point 1.

Solution

Since the function f has a pole of order four at the point 1, we apply Theorem 1.3 with $k=4$. We obtain

$$\begin{aligned}\operatorname{Res}(f,1) &= \frac{1}{3!}\lim_{z\to 1}\left(\frac{d^3}{dz^3}(1+e^{2z})\right)\\ &= \frac{1}{6}\lim_{z\to 1} 8e^{2z} = \frac{8}{6}e^2 = \frac{4}{3}e^2,\end{aligned}$$

in agreement with the solution to Exercise 1.7(b).

Exercise 1.8

Use Theorem 1.3 to find the residue of each of the following functions at the given point α.

(a) $f(z) = \dfrac{ze^{iz}}{(z-\pi)^2}, \quad \alpha = \pi$ (b) $f(z) = \dfrac{z+2}{z^3(z+4)}, \quad \alpha = 0$

Exercise 1.9

Prove Theorem 1.3.

Further exercises

Exercise 1.10

Find the residue of each of the following functions at the given points α.

(a) $f(z) = \dfrac{e^z}{z^7}, \quad \alpha = 0$ (b) $f(z) = \dfrac{\cos z}{(z-\pi/2)^2}, \quad \alpha = \pi/2$

(c) $f(z) = \dfrac{e^z}{z^4-1}, \quad \alpha = 1, -1, i, -i$ (d) $f(z) = \dfrac{1}{z^2-4}, \quad \alpha = 2, -2$

(e) $f(z) = \dfrac{e^z}{z^3(z^2-9)}, \quad \alpha = 0, 3, -3$

2 The Residue Theorem

After working through this section, you should be able to:

- state Cauchy's Residue Theorem, and use it to evaluate a given contour integral
- evaluate certain real trigonometric integrals of the form

$$\int_0^{2\pi} \Phi(\cos t, \sin t)\, dt,$$

  where Φ is a function of two variables.

2.1 Cauchy's Residue Theorem

At the end of Unit B4 you saw that if f is a function that is analytic on the punctured disc $D = \{z : 0 < |z-\alpha| < r\}$, then

$$\int_C f(z)\, dz = 2\pi i \operatorname{Res}(f, \alpha),$$

where C is any circle in D with centre α. It follows that we can evaluate such an integral by calculating the residue and multiplying the result by $2\pi i$.

We now extend this result to the case where f has several singularities at points inside a simple-closed contour Γ. In this case, each singularity α gives a contribution of $2\pi i \operatorname{Res}(f, \alpha)$ to the value of the integral; we then evaluate the integral of f along Γ by adding up these contributions, giving

$$2\pi i \times (\text{the sum of the residues at the singularities of } f \text{ inside } \Gamma).$$

The formal statement of this result is as follows; its proof is given in Subsection 2.3.

Theorem 2.1 Cauchy's Residue Theorem

Let $\mathcal{R}$ be a simply connected region, and let f be a function that is analytic on $\mathcal{R}$ except for a finite number of singularities. Let Γ be any simple-closed contour in $\mathcal{R}$, not passing through any of these singularities. Then

$$\int_\Gamma f(z)\,dz = 2\pi i S,$$

where S is the sum of the residues of f at those singularities that lie inside Γ.

Remarks

1. We usually refer to Theorem 2.1 simply as the 'Residue Theorem'.
2. Notice that if no singularities of f lie at points inside Γ, then the Residue Theorem reduces to a special case of Cauchy's Theorem, and the value of the integral is 0.
3. Following the convention from Subsection 1.3 of Unit B2, the contour Γ in the Residue Theorem is assumed to be traversed once anticlockwise.
4. When you use the Residue Theorem to evaluate an integral, it is good practice to draw the contour Γ and plot the singularities of the function f, for then it is easy to identify those singularities that lie *inside* Γ. If any of the singularities lie on Γ, then the integral is not defined. For example,

   $$\int_\Gamma \frac{1}{z(z-1)^2}\,dz,$$

   where $\Gamma = \{z : |z| = 1\}$, is not defined because the singularity 1 of $f(z) = 1/(z(z-1)^2)$ lies on Γ (see Figure 2.1).

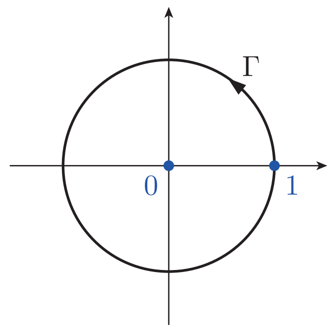


Figure 2.1 The point 1 lies on the unit circle

History of Cauchy's Residue Theorem

Cauchy first published his Residue Theorem in a paper entitled 'Mémoire sur les intégrales définies prises entre des limites imaginaires' (1825), considered by some to be his masterwork. At first, the full significance of the result was not recognised by mathematicians, including Cauchy himself, and it was only in the 1840s that its generality and the richness of its applications became properly appreciated.

The Residue Theorem can be used to evaluate integrals to which we applied the partial fractions strategy for evaluating contour integrals of Unit B2 (see, for instance, Example 2.1 below). However, it provides a much more general method, which applies to integrals (such as that of Exercise 2.4 below) to which the partial fractions strategy cannot be applied. The following examples illustrate the basic method.

Example 2.1

Evaluate the integral

$$\int_\Gamma \frac{1}{z(z-1)^2}\,dz,$$

where Γ is

(a) the circle $\left\{z : |z-1| = \frac{1}{2}\right\}$

(b) any rectangular contour containing both 0 and 1 in its inside.

Solution

(a) The function

$$f(z) = \frac{1}{z(z-1)^2}$$

is analytic on $\mathbb{C}$ apart from singularities at 0 and 1. The point 0 lies outside $\Gamma = \left\{z : |z-1| = \frac{1}{2}\right\}$, and 1 lies inside Γ (see Figure 2.2). By Example 1.2(b), $\operatorname{Res}(f,1) = -1$.

Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$,

$$\int_\Gamma f(z)\,dz = 2\pi i \operatorname{Res}(f,1) = -2\pi i.$$

(b) Both singularities 0 and 1 of f lie inside Γ (see Figure 2.3). As in part (a), $\operatorname{Res}(f,1) = -1$, and by Example 1.2(a) or the Cover-up Rule, $\operatorname{Res}(f,0) = 1$.

Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$,

$$\begin{aligned}\int_\Gamma f(z)\,dz &= 2\pi i(\operatorname{Res}(f,1) + \operatorname{Res}(f,0))\\ &= 2\pi i(-1+1) = 0.\end{aligned}$$

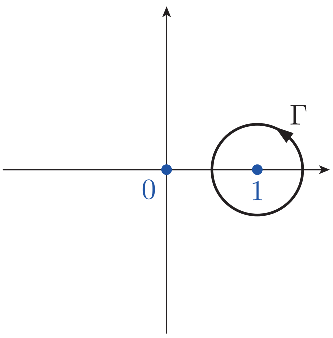


Figure 2.2 The circle $\Gamma = \left\{z : |z-1| = \frac{1}{2}\right\}$

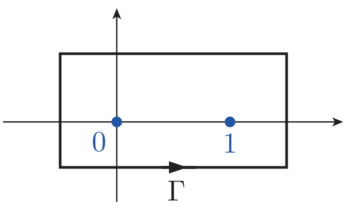


Figure 2.3 A rectangular contour containing both 0 and 1

Example 2.2

Evaluate the integral

$$\int_\Gamma \frac{z+3}{(z-1)(z-2)(z+4)}\,dz,$$

where Γ is the ellipse $\{z = x + iy : 4x^2 + 9y^2 = 36\}$.

Solution

The function

$$f(z) = \frac{z+3}{(z-1)(z-2)(z+4)}$$

is analytic on $\mathbb{C}$ apart from simple poles at 1, 2 and -4. Of these, 1 and 2 lie inside $\Gamma = \{z = x + iy : 4x^2 + 9y^2 = 36\}$, and -4 lies outside Γ (see Figure 2.4).

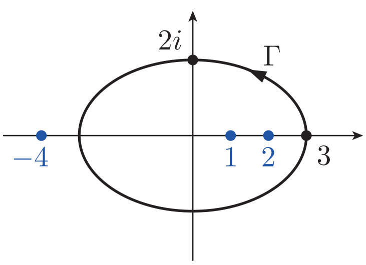


Figure 2.4 The ellipse $\Gamma = \{z = x + iy : 4x^2 + 9y^2 = 36\}$

By the Cover-up Rule,

$$\operatorname{Res}(f,1) = \frac{1+3}{(1-2)(1+4)} = -\frac{4}{5},$$
$$\operatorname{Res}(f,2) = \frac{2+3}{(2-1)(2+4)} = \frac{5}{6}.$$

Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$,

$$\begin{aligned}\int_\Gamma f(z)\,dz &= 2\pi i(\operatorname{Res}(f,1) + \operatorname{Res}(f,2))\\ &= 2\pi i\left(-\frac{4}{5} + \frac{5}{6}\right) = \frac{\pi i}{15}.\end{aligned}$$

Observe that although the function f from Example 2.2 has a pole at -4, we do not need to calculate the residue of f at -4 to apply the Residue Theorem because -4 lies *outside* the contour Γ.

Exercise 2.1

Evaluate the integral

$$\int_\Gamma \frac{\sin z}{z^2 - 1}\,dz,$$

where Γ is

(a) the circle $\{z : |z| = 3\}$

(b) the rectangular contour with vertices at $-2i$, $2i$, $-2+2i$, $-2-2i$.

Exercise 2.2

Let Γ be the circle $\{z : |z - i| = 2\}$. Evaluate the integral

$$I = \int_\Gamma \frac{z+2}{4z^2 + k^2}\,dz,$$

for each of the following values of k.

(a) $k = 1$ (b) $k = 3$ (c) $k = 7$

Exercise 2.3

Use the result of Exercise 1.4(c) to evaluate the integral

$$\int_\Gamma \frac{z^3}{z^4 + 1}\,dz,$$

where Γ is the semicircular contour shown in Figure 2.5.

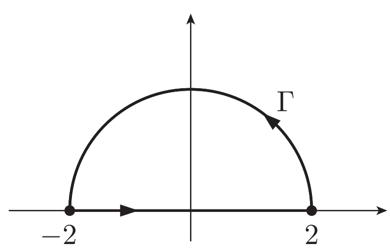


Figure 2.5 A simple-closed semicircular contour

Exercise 2.4

Evaluate the integral

$$\int_\Gamma \frac{1+z}{\sin z}\, dz,$$

where Γ is the square contour with vertices $4+4i$, $-4+4i$, $-4-4i$, $4-4i$.

(*Hint*: Choose the region $\mathcal{R}$ carefully so that it contains only a finite number of singularities of the function.)

2.2 Real trigonometric integrals

The Residue Theorem proves to be useful in some quite unexpected ways. One of these is in the evaluation of real integrals of the form

$$\int_0^{2\pi} \Phi(\cos t, \sin t)\, dt,$$

where Φ is a function of two real variables. For example, if

$$\Phi(x,y) = \frac{1}{5+4y},$$

then

$$\int_0^{2\pi} \Phi(\cos t, \sin t)\, dt = \int_0^{2\pi} \frac{1}{5+4\sin t}\, dt.$$

Let us prove that this integral is equal to the contour integral

$$\int_C \frac{1}{2z^2+5iz-2}\, dz,$$

where C is the unit circle $\{z : |z| = 1\}$. Using the parametrisation

$$\gamma(t) = e^{it} \quad (t \in [0, 2\pi]),$$

and observing that $\gamma'(t) = ie^{it}$ and $\sin t = (e^{it} - e^{-it})/(2i)$, we obtain

$$\begin{aligned}
\int_C \frac{1}{2z^2+5iz-2}\, dz &= \int_0^{2\pi} \frac{ie^{it}}{2e^{2it}+5ie^{it}-2}\, dt \\
&= \int_0^{2\pi} \frac{i}{2e^{it}+5i-2e^{-it}}\, dt \\
&= \int_0^{2\pi} \frac{1}{5+2(e^{it}-e^{-it})/i}\, dt \\
&= \int_0^{2\pi} \frac{1}{5+4\sin t}\, dt.
\end{aligned}$$

This argument can be reversed to transform the given real trigonometric integral into a contour integral around the unit circle C.

Since choosing $\gamma(t) = e^{it}$ amounts to substituting $z = e^{it}$, we need to replace

$$\cos t \text{ by } \frac{1}{2}\left(z + z^{-1}\right), \quad \text{since } \cos t = \frac{1}{2}(e^{it} + e^{-it}) = \frac{1}{2}\left(z + z^{-1}\right),$$

$$\sin t \text{ by } \frac{1}{2i}\left(z - z^{-1}\right), \quad \text{since } \sin t = \frac{1}{2i}(e^{it} - e^{-it}) = \frac{1}{2i}\left(z - z^{-1}\right),$$

$$dt \text{ by } \frac{1}{iz}\,dz, \quad \text{since } \frac{dz}{dt} = ie^{it} = iz.$$

Using this procedure, we can replace a trigonometric integral $\int_0^{2\pi} \Phi(\cos t, \sin t)\,dt$ by a contour integral of the form $\int_C f(z)\,dz$, which we can evaluate using the Residue Theorem. An example will make the method clear.

Example 2.3

Evaluate the integral

$$\int_0^{2\pi} \frac{1}{5 + 4\sin t}\,dt.$$

Solution

If C is the unit circle $\{z : |z| = 1\}$, then, following the discussion above,

$$\begin{aligned}\int_0^{2\pi} \frac{1}{5 + 4\sin t}\,dt &= \int_C \frac{1}{5 + 4(z - z^{-1})/(2i)} \times \frac{1}{iz}\,dz \\ &= \int_C \frac{1}{5iz + 2(z^2 - 1)}\,dz \\ &= \int_C \frac{1}{2z^2 + 5iz - 2}\,dz,\end{aligned}$$

as expected. Now,

$$2z^2 + 5iz - 2 = (2z + i)(z + 2i) = 2\left(z + \tfrac{1}{2}i\right)(z + 2i),$$

so the singularities of the function $f(z) = 1/(2z^2 + 5iz - 2)$ are simple poles at $-\frac{1}{2}i$ and $-2i$. The pole $-\frac{1}{2}i$ lies inside the unit circle C, and the pole $-2i$ lies outside C.

By the Cover-up Rule,

$$\operatorname{Res}\left(f, -\tfrac{1}{2}i\right) = \frac{1}{2\left(-\frac{1}{2}i + 2i\right)} = \frac{1}{3i}.$$

It follows from the Residue Theorem with $\mathcal{R} = \mathbb{C}$ that

$$\int_0^{2\pi} \frac{1}{5 + 4\sin t}\,dt = 2\pi i \times \frac{1}{3i} = \frac{2\pi}{3}.$$

Note that the answer is a real number, as it should be, because it is the value of the integral of a real function.

We can summarise the method above in the form of a strategy.

Strategy for evaluating real trigonometric integrals

To evaluate a real integral of the form

$$\int_0^{2\pi} \Phi(\cos t, \sin t)\, dt,$$

where Φ is a function of two real variables, proceed as follows.

1. Replace

$$\cos t \text{ by } \frac{1}{2}\left(z + z^{-1}\right), \quad \sin t \text{ by } \frac{1}{2i}\left(z - z^{-1}\right), \quad dt \text{ by } \frac{1}{iz}\, dz,$$

   to obtain a contour integral of the form $\int_C f(z)\, dz$ around the unit circle $C = \{z : |z| = 1\}$. In order for the strategy to apply, the function f must be analytic with finitely many singularities on a simply connected region that contains C, and none of the singularities can lie on C.
2. Locate the singularities of the function f lying inside C, and calculate the residues of f at these points.
3. Evaluate the given integral by calculating

$$2\pi i \times (\text{the sum of the residues found in step 2}).$$

Exercise 2.5

Evaluate the integral

$$\int_0^{2\pi} \frac{1}{2 - \cos t}\, dt.$$

The next exercise makes use of the binomial coefficient $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$.

Exercise 2.6

Let n be a positive integer.

(a) Show that the residue of the function $f(z) = \left(z^2+1\right)^n / z^{n+1}$ at the point 0 is

$$\binom{n}{\frac{1}{2}n} \text{ if } n \text{ is even, and } 0 \text{ if } n \text{ is odd.}$$

(b) Use this result to evaluate the integral

$$\int_0^{2\pi} \cos^n t\, dt.$$

2.3 Proof of the Residue Theorem

We now sketch a proof of the Residue Theorem. The proof is challenging, so you may consider just skimming through the details on a first reading.

Theorem 2.1 Cauchy's Residue Theorem

Let $\mathcal{R}$ be a simply connected region, and let f be a function that is analytic on $\mathcal{R}$ except for a finite number of singularities. Let Γ be any simple-closed contour in $\mathcal{R}$, not passing through any of these singularities. Then

$$\int_\Gamma f(z)\,dz = 2\pi i S,$$

where S is the sum of the residues of f at those singularities that lie inside Γ.

Proof Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the singularities of f lying *inside* Γ. For each singularity α_k, choose a circle C_k inside Γ that is centred at α_k and has no other singularities of f on or inside it. Choose the circles $C_1, C_2, \ldots, C_n$ in such a way that they do not meet one another, as illustrated in Figure 2.6(a) for the case $n = 3$. Since f is analytic on a punctured open disc centred at α_k containing C_k, we have, by equation (4.3) of Unit B4,

$$\int_{C_k} f(z)\,dz = 2\pi i \operatorname{Res}(f, \alpha_k), \quad \text{for } k = 1, 2, \ldots, n. \tag{2.1}$$

We wish to show that

$$\int_\Gamma f(z)\,dz = \sum_{k=1}^{n}\left(\int_{C_k} f(z)\,dz\right).$$

To do this, we express $\int_\Gamma f(z)\,dz$ in terms of integrals around $C_1, C_2, \ldots, C_n$ and around other simple-closed contours, where the integral of f around each of these other contours is zero (by Cauchy's Theorem).

For each $k = 1, 2, \ldots, n$, join C_k to Γ by two simple smooth paths L_k and L'_k in such a way that no two of the contours $L_1, L_2, \ldots, L_n, L'_1, L'_2, \ldots, L'_n$ meet, as indicated in Figure 2.6(b) for the case $n = 3$.

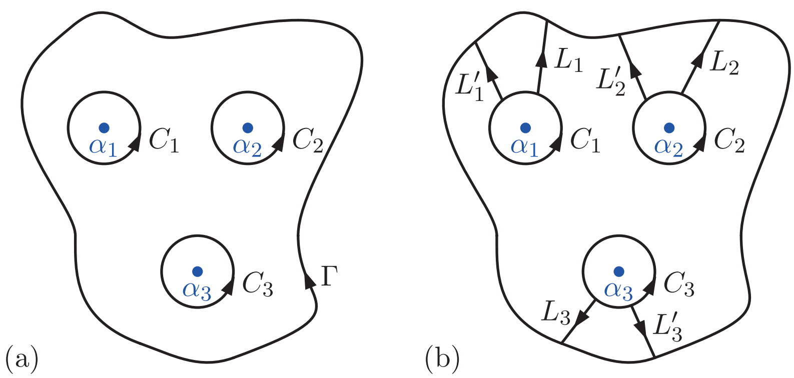


Figure 2.6 (a) Circles C_k inside Γ centred at α_k (b) Smooth paths L_k and L'_k joining C_k to Γ

We choose these smooth paths such that, for each k, the paths L_k and $\widetilde{L}'_k$ ($\widetilde{L}'_k$ is the reverse path of L'_k) form part of a simple-closed contour Γ_k (illustrated in Figure 2.7(a), for $k = 1, 2, 3$) that contains no singularities of f.

Each contour Γ_k contains part of $\widetilde{C}_k$ (the reverse contour of C_k), part of Γ, and the simple contours L_k and $\widetilde{L}'_k$. The contour Γ', illustrated in Figure 2.7(b), contains the left-over parts of Γ, and, for each k, it contains both $\widetilde{L}_k$ and L'_k, as well as the remaining part of $\widetilde{C}_k$ that is not part of Γ_k.

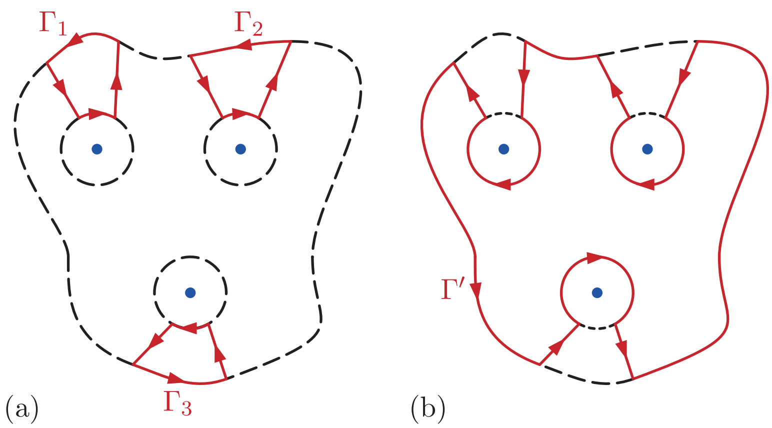


Figure 2.7 Simple-closed contours (a) $\Gamma_1, \Gamma_2, \Gamma_3$ and (b) Γ'

If we add the integral of f along Γ' to the sum of the integrals of f along the contours Γ_k, then the parts of these integrals along the simple contours L_k and $\widetilde{L}_k$, and L'_k and $\widetilde{L}'_k$, cancel. Consequently, we obtain

$$\int_{\Gamma'} f(z)\,dz + \sum_{k=1}^{n}\left(\int_{\Gamma_k} f(z)\,dz\right) = \int_{\Gamma} f(z)\,dz + \sum_{k=1}^{n}\left(\int_{\widetilde{C}_k} f(z)\,dz\right).$$

But, by Cauchy's Theorem (applied to Γ' and to each Γ_k on appropriate simply connected regions),

$$\int_{\Gamma'} f(z)\,dz = 0 \quad \text{and} \quad \int_{\Gamma_k} f(z)\,dz = 0, \quad \text{for } k = 1, 2, \ldots, n.$$

Hence

$$\begin{aligned}\int_{\Gamma} f(z)\,dz &= -\sum_{k=1}^{n}\left(\int_{\widetilde{C}_k} f(z)\,dz\right)\\ &= \sum_{k=1}^{n}\left(\int_{C_k} f(z)\,dz\right)\\ &= \sum_{k=1}^{n} 2\pi i\,\mathrm{Res}(f, \alpha_k) \quad \text{(by equation (2.1))}\\ &= 2\pi i \sum_{k=1}^{n}\mathrm{Res}(f, \alpha_k).\end{aligned}$$

■

Further exercises

Exercise 2.7

Use the Residue Theorem to evaluate each of the following contour integrals around the circle $C = \{z : |z| = 2\}$.

(a) $\displaystyle\int_C \frac{\cos z}{(z-\pi/2)^2}\,dz$ (b) $\displaystyle\int_C \frac{e^z}{z^4-1}\,dz$ (c) $\displaystyle\int_C \frac{e^z}{z^3(z^2-9)}\,dz$

(*Hint*: Use results from Exercise 1.10.)

Exercise 2.8

Use the Residue Theorem to evaluate

$$\int_C \frac{z}{e^z-1}\,dz,$$

for each of the following contours C.

(a) $C = \{z : |z| = 1\}$ (b) $C = \{z : |z-3i| = 4\}$

Exercise 2.9

Show that

$$\int_0^{2\pi} \frac{1}{16\cos^2 t + 9}\,dt = \frac{1}{4i}\int_C \frac{z}{(z^2+4)\left(z^2+\frac{1}{4}\right)}\,dz,$$

where C is the unit circle $\{z : |z| = 1\}$.

Hence evaluate the given trigonometric integral.

3 Evaluating improper integrals

After working through this section, you should be able to:

- understand the definitions of improper real integrals
- use the Residue Theorem to evaluate improper real integrals of the forms

  $$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\,dt, \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\cos kt\,dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\sin kt\,dt,$$

  where p and q are real polynomial functions subject to suitable restrictions, and k is a real constant.

3.1 Improper integrals

Integrals of the forms

$$\int_{-\infty}^{\infty} f(t)\,dt \quad \text{and} \quad \int_{0}^{\infty} f(t)\,dt,$$

where f is a real-valued or complex-valued function of the real variable t, occur in many branches of mathematics. They are called *improper integrals*. For example, the improper integral

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt,$$

sometimes called the *Gaussian integral*, is important in statistics because of its connection with the normal distribution, and the *Fourier transform* of a function $f\colon \mathbb{R} \longrightarrow \mathbb{C}$, defined by

$$\widetilde{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt}\,dx,$$

arises in the theory of differential equations. (We will meet Fourier transforms again at the end of this section, and we will meet the Gaussian integral again in Section 6 of Unit C2.)

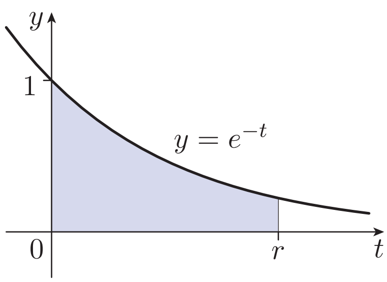


Figure 3.1 Area under the graph of $y = e^{-t}$ between 0 and r

Our aim in this subsection is to define such improper integrals. The idea of the definition is to consider an ordinary integral such as

$$\int_{0}^{r} e^{-t}\,dt,$$

which represents the shaded area in Figure 3.1, and then *define*

$$\int_{0}^{\infty} e^{-t}\,dt = \lim_{r\to\infty} \int_{0}^{r} e^{-t}\,dt,$$

provided that this limit exists.

First, however, we need to define the notion of a limit as r tends to ∞.

Definition

Let f be a function defined on an unbounded interval (a, ∞), and suppose that $\alpha \in \mathbb{C}$. The function f has **limit α as r tends to ∞** if for each real sequence (r_n) in (a, ∞) such that $r_n \to \infty$ as $n \to \infty$, we have

$$f(r_n) \to \alpha \text{ as } n \to \infty.$$

In this case we write either

$$\lim_{r\to\infty} f(r) = \alpha \quad \text{or} \quad f(r) \to \alpha \text{ as } r \to \infty.$$

There is an equivalent ε-N definition of this limit, which says that for each positive number ε, there is an integer N such that

$$|f(r) - \alpha| < \varepsilon, \quad \text{for all } r > N.$$

The next example shows how to evaluate a limit using the sequence definition.

Example 3.1

Evaluate the limit

$$\lim_{r\to\infty} \frac{1}{r^2}.$$

Solution

Intuitively, we suspect that the limit is 0. Let

$$f(r) = \frac{1}{r^2} \quad (r \in (0,\infty)).$$

If (r_n) is a sequence in $(0,\infty)$ such that $r_n \to \infty$, then $1/r_n \to 0$ by the Reciprocal Rule for sequences (Theorem 1.5 of Unit A3), and hence $1/r_n^2 \to 0$, by the Product Rule for sequences (Theorem 1.3(c) of Unit A3). Therefore

$$f(r) = 1/r^2 \to 0 \text{ as } r \to \infty.$$

Exercise 3.1

Use the ε-N definition to prove that

$$\lim_{r\to\infty} \frac{1}{\sqrt{r}} = 0.$$

More complicated limits can be evaluated by using the following theorem, which follows from the Combination Rules for sequences. We omit the proof.

Theorem 3.1 Combination Rules for Limits of Functions

Let f and g be functions such that

$$\lim_{r\to\infty} f(r) = \alpha \quad \text{and} \quad \lim_{r\to\infty} g(r) = \beta.$$

(a) **Sum Rule** $\lim_{r\to\infty} (f(r) + g(r)) = \alpha + \beta.$

(b) **Multiple Rule** $\lim_{r\to\infty} (\lambda f(r)) = \lambda\alpha,$ for $\lambda \in \mathbb{C}.$

(c) **Product Rule** $\lim_{r\to\infty} (f(r)g(r)) = \alpha\beta.$

(d) **Quotient Rule** $\lim_{r\to\infty} (f(r)/g(r)) = \alpha/\beta,$ provided that $\beta \neq 0.$

We will need the limit calculated in the next example later in the section.

Example 3.2

Evaluate the limit

$$\lim_{r\to\infty} \frac{\pi r}{r^2 - 1}.$$

Solution

It seems likely that the limit is 0, since the power of r occurring in the denominator is larger than that occurring in the numerator.

To prove that the limit is indeed 0, we divide the numerator and denominator by the 'dominant term' r^2, and write

$$\lim_{r\to\infty} \frac{\pi r}{r^2 - 1} = \lim_{r\to\infty} \frac{\pi/r}{1 - (1/r^2)}.$$

But

$$\lim_{r\to\infty} 1/r = 0 \quad \text{and} \quad \lim_{r\to\infty} 1/r^2 = 0.$$

It follows from the Sum, Multiple and Quotient Rules that

$$\lim_{r\to\infty} \frac{\pi/r}{1 - (1/r^2)} = \frac{\pi \times 0}{1 - 0} = 0.$$

The limit evaluated in Example 3.2 is a special case of the following useful corollary to Theorem 3.1 (which we often use without explicit reference).

Corollary

If p and q are polynomial functions such that the degree of q exceeds the degree of p, then

$$\lim_{r\to\infty} \frac{p(r)}{q(r)} = 0.$$

Exercise 3.2

Prove the corollary to Theorem 3.1.

We can now define the improper integrals

$$\int_{-\infty}^{\infty} f(t)\,dt \quad \text{and} \quad \int_{a}^{\infty} f(t)\,dt.$$

Definitions

Let f be a function that is continuous on $\mathbb{R}$. Then the **improper integral** $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt$ is

$$\int_{-\infty}^{\infty} f(t)\,dt = \lim_{r\to\infty} \int_{-r}^{r} f(t)\,dt,$$

provided that this limit exists.

Let f be a function that is continuous on the interval $[a,\infty)$. Then the **improper integral** $\displaystyle\int_{a}^{\infty} f(t)\,dt$ is

$$\int_{a}^{\infty} f(t)\,dt = \lim_{r\to\infty} \int_{a}^{r} f(t)\,dt,$$

provided that this limit exists.

Remark

An alternative, but *not* equivalent, definition of the improper integral

$$\int_{-\infty}^{\infty} f(t)\,dt$$

is

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{\infty} f(t)\,dt, \tag{3.1}$$

where $\displaystyle\int_{0}^{\infty} f(t)\,dt$ is defined as above and

$$\int_{-\infty}^{0} f(t)\,dt = \lim_{r\to\infty} \int_{-r}^{0} f(t)\,dt,$$

provided that this limit exists.

If the integrals on the right-hand side of equation (3.1) exist, then the two definitions of $\displaystyle\int_{-\infty}^{\infty} f(t)\,dt$ give the same value. However, this does not always happen; for example, the integral

$$\int_{-\infty}^{\infty} t\,dt$$

defined according to equation (3.1) does *not* exist because

$$\int_{0}^{r} t\,dt = \frac{r^2}{2},$$

and $r^2/2$ does not tend to a limit as r tends to ∞.

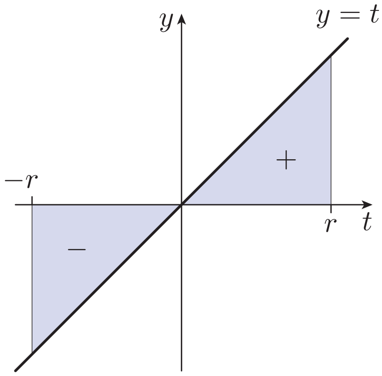


Figure 3.2 Signed area bounded by the graph of $y = t$ between $-r$ and r

In contrast, using our definition, we see that

$$\int_{-r}^{r} t\,dt = \frac{r^2}{2} - \frac{r^2}{2} = 0,$$

as illustrated in Figure 3.2 (the positive and negative areas cancel). Hence

$$\int_{-\infty}^{\infty} t\,dt = \lim_{r\to\infty} \int_{-r}^{r} t\,dt$$

does exist, and it equals 0.

Some texts that do not adopt our definition refer to our improper integral as the *principal value of the integral*.

Example 3.3

Evaluate the following improper integrals.

(a) $\displaystyle\int_{-\infty}^{\infty} t^3\,dt$ (b) $\displaystyle\int_{0}^{\infty} \frac{1}{t^2+1}\,dt$

Solution

(a) $$\int_{-\infty}^{\infty} t^3\,dt = \lim_{r\to\infty} \int_{-r}^{r} t^3\,dt = \lim_{r\to\infty} \left[\tfrac{1}{4}t^4\right]_{-r}^{r} = \lim_{r\to\infty} 0 = 0.$$

(b) $$\int_{0}^{\infty} \frac{1}{t^2+1}\,dt = \lim_{r\to\infty} \int_{0}^{r} \frac{1}{t^2+1}\,dt = \lim_{r\to\infty} \left[\tan^{-1} t\right]_{0}^{r} = \lim_{r\to\infty} \tan^{-1} r = \pi/2.$$

See Figure 3.3 for the graph of $\tan^{-1}$.

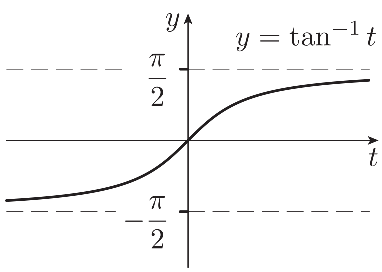


Figure 3.3 Graph of $y = \tan^{-1} t$

Exercise 3.3

Evaluate each of the following improper integrals.

(a) $\displaystyle\int_{-\infty}^{\infty} \sin t\,dt$ (b) $\displaystyle\int_{1}^{\infty} \frac{1}{t^p}\,dt$, where $p > 1$ (c) $\displaystyle\int_{0}^{\infty} e^{-t}\,dt$

The following result is useful for evaluating improper real integrals of odd and even functions.

Theorem 3.2

Let f be a function that is continuous on $\mathbb{R}$.

(a) If f is an odd function, then

$$\int_{-\infty}^{\infty} f(t)\,dt = 0.$$

(b) If f is an even function, then

$$\int_{-\infty}^{\infty} f(t)\,dt = 2\int_{0}^{\infty} f(t)\,dt,$$

provided that these improper integrals exist.

We prove part (a), leaving you to prove part (b) in Exercise 3.4.

Proof If f is an odd function, then

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\,dt &= \lim_{r\to\infty}\int_{-r}^{r} f(t)\,dt\\ &= \lim_{r\to\infty}\left(\int_{0}^{r} f(t)\,dt + \int_{-r}^{0} f(t)\,dt\right).\end{aligned}$$

Making the change of variables $u = -t$, so $\dfrac{du}{dt} = -1$, in the second integral, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\,dt &= \lim_{r\to\infty}\left(\int_{0}^{r} f(t)\,dt + \int_{r}^{0} f(-u)(-du)\right)\\ &= \lim_{r\to\infty}\left(\int_{0}^{r} f(t)\,dt - \int_{0}^{r} f(u)\,du\right)\\ &= \lim_{r\to\infty} 0 = 0,\end{aligned}$$

where we have used the fact that $f(-u) = -f(u)$, because f is an odd function. This proves part (a). ■

Exercise 3.4

Prove Theorem 3.2(b).

We also need two other types of improper integral. They both involve real functions that are continuous on some interval except at one point, such as $f(t) = 1/t$.

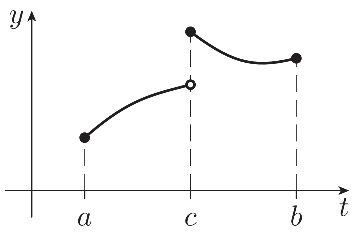


Figure 3.4 Graph of a function that is defined at c, but discontinuous at c

Definitions

Let f be a function that is continuous at all points of an interval $[a, b]$ except the point $c \in (a, b)$, at which f may or may not be defined (see Figure 3.4). Then the **improper integral** $\int_a^b f(t)\,dt$ is

$$\int_a^b f(t)\,dt = \lim_{\varepsilon \to 0}\left(\int_a^{c-\varepsilon} f(t)\,dt + \int_{c+\varepsilon}^b f(t)\,dt\right),$$

provided that this limit, which is taken through *positive* values of ε, exists.

Let f be a function that is continuous at all points of $\mathbb{R}$ except the point c. Then the **improper integral** $\int_{-\infty}^{\infty} f(t)\,dt$ is

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\,dt &= \lim_{r \to \infty} \int_{-r}^{r} f(t)\,dt \\ &= \lim_{r \to \infty}\left(\lim_{\varepsilon \to 0}\left(\int_{-r}^{c-\varepsilon} f(t)\,dt + \int_{c+\varepsilon}^{r} f(t)\,dt\right)\right),\end{aligned}$$

provided that these limits exist.

Remarks

1. The definitions above can easily be extended to functions that are continuous except at a finite number of points.
2. Throughout this unit the expression $\lim\limits_{\varepsilon \to 0}$ always represents a limit taken through *positive* values of ε.

Example 3.4

Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{t^3}\,dt.$$

Solution

The function $f(t) = 1/t^3$ is continuous at all points of $\mathbb{R}$ except at the point 0 (see Figure 3.5). Thus, for $r > 0$,

$$\begin{aligned}\int_{-r}^{r} \frac{1}{t^3}\,dt &= \lim_{\varepsilon \to 0}\left(\int_{-r}^{-\varepsilon} \frac{1}{t^3}\,dt + \int_{\varepsilon}^{r} \frac{1}{t^3}\,dt\right) \\ &= \lim_{\varepsilon \to 0}\left(\left[-\frac{1}{2t^2}\right]_{-r}^{-\varepsilon} + \left[-\frac{1}{2t^2}\right]_{\varepsilon}^{r}\right) \\ &= \lim_{\varepsilon \to 0}\left(-\frac{1}{2\varepsilon^2} + \frac{1}{2r^2} - \frac{1}{2r^2} + \frac{1}{2\varepsilon^2}\right) \\ &= \lim_{\varepsilon \to 0} 0 = 0.\end{aligned}$$

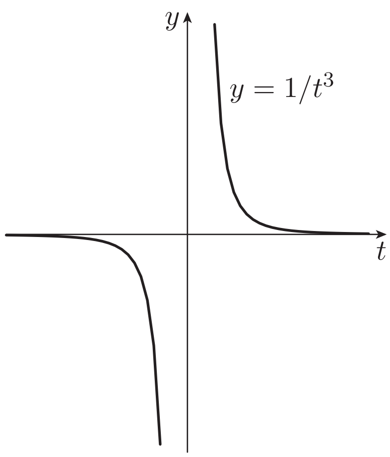


Figure 3.5 Graph of $y = 1/t^3$

Hence

$$\int_{-\infty}^{\infty} \frac{1}{t^3}\, dt = \lim_{r\to\infty} \int_{-r}^{r} \frac{1}{t^3}\, dt = \lim_{r\to\infty} 0 = 0.$$

This answer is to be expected because $f(t) = 1/t^3$ is an odd function.

Exercise 3.5

Evaluate the improper integral

$$\int_{-1}^{2} \frac{1}{t}\, dt.$$

3.2 Methods for integrating rational functions

In this subsection we show how the Residue Theorem can be used to evaluate a whole class of real improper integrals of the forms

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\, dt, \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \cos kt\, dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \sin kt\, dt,$$

where p and q are real polynomial functions subject to suitable restrictions, and k is a real constant.

To illustrate the procedure, we evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{t^2+1}\, dt,$$

which, by definition, is equal to

$$\lim_{r\to\infty} \int_{-r}^{r} \frac{1}{t^2+1}\, dt.$$

By using methods from real analysis, we saw in Example 3.3(b) that

$$\int_{0}^{\infty} \frac{1}{t^2+1}\, dt = \frac{\pi}{2},$$

and since $t \longmapsto 1/(t^2+1)$ is an even function, it follows from Theorem 3.2(b) that

$$\int_{-\infty}^{\infty} \frac{1}{t^2+1}\, dt = \pi.$$

Let us now obtain the same result by applying the Residue Theorem. The solution is broken down into five steps to help explain the procedure.

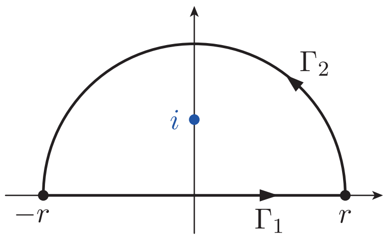


Figure 3.6 A simple-closed semicircular contour

1. Find a suitable contour integral

The first step is to identify a contour integral of the *complex* expression $1/(z^2+1)$, which we will evaluate to give us the required integral of the *real* expression $1/(t^2+1)$. We choose the contour integral

$$I = \int_\Gamma \frac{1}{z^2+1}\,dz,$$

where $\Gamma = \Gamma_1 + \Gamma_2$ is the contour shown in Figure 3.6, traversed anticlockwise. Here Γ_1 is the interval $[-r, r]$ on the real axis, and Γ_2 is the semicircle centred at 0 of radius r that lies in the upper half-plane.

The integrand of the contour integral I is the complex function

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}.$$

This function is analytic on the simply connected region $\mathbb{C}$ except for simple poles at i and $-i$. Let us choose our contour Γ such that $r > 1$ so that the pole at i lies inside Γ (and $-i$ lies outside Γ).

2. Apply the Residue Theorem

To apply the Residue Theorem, we need to find the residue of f at i. By the Cover-up Rule, it is

$$\operatorname{Res}(f, i) = \frac{1}{i+i} = \frac{1}{2i}.$$

We can now apply the Residue Theorem to give

$$I = 2\pi i \operatorname{Res}(f, i) = 2\pi i \times \frac{1}{2i} = \pi.$$

Notice that this answer is independent of the radius r of the semicircle, provided that $r > 1$. This fact is crucial in step 5 when we let $r \to \infty$.

3. Split up the integral

The next step is to split the integral into its constituent parts to give

$$I = \int_{\Gamma_1} \frac{1}{z^2+1}\,dz + \int_{\Gamma_2} \frac{1}{z^2+1}\,dz. \tag{3.2}$$

Since Γ_1 is a real interval, we can write the first integral as a real integral: we simply substitute t for z and dt for dz, and put in the limits of integration $t = -r$ and $t = r$, to give

$$\int_{\Gamma_1} \frac{1}{z^2+1}\,dz = \int_{-r}^{r} \frac{1}{t^2+1}\,dt.$$

This integral is now similar to the original integral that we were asked to evaluate; the only difference is that this integral has limits $-r$ and r rather than $-\infty$ and ∞. In the next two steps we will show that the Γ_2 integral tends to 0 as $r \to \infty$, which will allow us to evaluate the improper integral using the value of I found in step 2.

4. Apply the Estimation Theorem

Let us apply the Estimation Theorem (Theorem 4.1 of Unit B1) to obtain an upper estimate for

$$\left|\int_{\Gamma_2} \frac{1}{z^2+1}\,dz\right|.$$

To do this, we need an upper estimate for $|f(z)| = 1/|z^2+1|$ on Γ_2, which is equivalent to a *lower* estimate for $|z^2+1|$ on Γ_2. Notice that $|z| = r$, for $z \in \Gamma_2$. Using the backwards form of the Triangle Inequality, we see that for $z \in \Gamma_2$,

$$|z^2+1| \geq |z^2| - |1| = |z|^2 - 1 = r^2 - 1.$$

Taking the reciprocal of both sides of the inequality $|z^2+1| \geq r^2-1$, we obtain $|f(z)| \leq M$, where $M = 1/(r^2-1)$. The length L of Γ_2 is πr (half the circumference of the circle $\{z : |z| = r\}$). Since f is continuous on Γ_2, we can apply the Estimation Theorem to give

$$\left|\int_{\Gamma_2} \frac{1}{z^2+1}\,dz\right| \leq ML = \frac{\pi r}{r^2-1},$$

for $r > 1$.

5. Take limits

From Example 3.2, we know that $\pi r/(r^2-1) \to 0$ as $r \to \infty$, so we see that for any positive number ε, there is an integer N such that

$$\left|\int_{\Gamma_2} \frac{1}{z^2+1}\,dz\right| \leq \left|\frac{\pi r}{r^2-1}\right| < \varepsilon,$$

for all $r > N$. Hence

$$\lim_{r\to\infty} \int_{\Gamma_2} \frac{1}{z^2+1}\,dz = 0.$$

Recall that $I = \pi$, by step 2. So equation (3.2) can be rearranged to give

$$\int_{-r}^{r} \frac{1}{t^2+1}\,dt = \pi - \int_{\Gamma_2} \frac{1}{z^2+1}\,dz.$$

Hence

$$\lim_{r\to\infty} \int_{-r}^{r} \frac{1}{t^2+1}\,dt = \lim_{r\to\infty}\left(\pi - \int_{\Gamma_2} \frac{1}{z^2+1}\,dz\right) = \pi.$$

That is, the limit of $\displaystyle\int_{-r}^{r} \frac{1}{t^2+1}\,dt$ as $r \to \infty$ exists, and has value π. So

$$\int_{-\infty}^{\infty} \frac{1}{t^2+1}\,dt = \pi,$$

as expected.

We will often apply this five-step strategy in future examples and exercises, although we condense the reasoning somewhat. In particular, sometimes in the final step we evaluate the improper integral without first proving that it exists, because the proof of existence follows implicitly from the argument that determines its value.

Try the five-step strategy in the following exercise.

Exercise 3.6

Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{1}{t^4+1}\,dt.$$

Let us consider another example, the improper integral

$$\int_{-\infty}^{\infty} \frac{t}{(t^2+1)(t-2)}\,dt,$$

which cannot be evaluated using techniques from real analysis as easily as the previous example.

To apply the earlier strategy, we must use the complex function

$$f(z) = \frac{z}{(z^2+1)(z-2)}.$$

However, there is a problem because f has a singularity at 2, so we cannot use the same contour as before when we apply the Residue Theorem (because that contour crossed the point 2 when $r > 2$). Instead, we can take a 'detour' to avoid this singularity, using the following lemma.

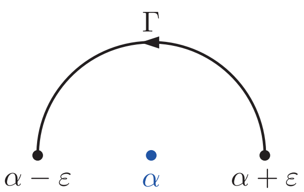


Figure 3.7 Semicircular contour

Lemma 3.1 Round-the-Pole Lemma

Suppose that f is a function that is analytic on a punctured disc $\{z : 0 < |z - \alpha| < \delta\}$ and has a simple pole at α. Let Γ be the upper half of the circle centred at α of radius ε, where $\varepsilon < \delta$, traversed from $\alpha + \varepsilon$ to $\alpha - \varepsilon$. Then

$$\lim_{\varepsilon \to 0} \int_{\Gamma} f(z)\,dz = \pi i \operatorname{Res}(f, \alpha).$$

The path Γ is illustrated in Figure 3.7.

Informally speaking, we can think of the lemma as 'half the Residue Theorem' (for a function f with a single, simple pole), because the semicircular path is half of a circular path, and the value of the limit is half of $2\pi i \operatorname{Res}(f, \alpha)$.

We defer the proof of the lemma until the next subsection, and instead proceed with the promised example.

Example 3.5

Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{t}{(t^2+1)(t-2)}\,dt.$$

Solution

The integrand

$$f(t) = \frac{t}{(t^2+1)(t-2)}$$

is not defined at the point 2 (but is defined everywhere else on the real axis). So, using the definitions of improper integrals from the previous subsection, we see that

$$\int_{-\infty}^{\infty} f(t)\,dt = \lim_{r\to\infty}\left(\lim_{\varepsilon\to 0}\left(\int_{-r}^{2-\varepsilon} f(t)\,dt + \int_{2-\varepsilon}^{r} f(t)\,dt\right)\right),$$

provided that these limits exist.

1. Let us now think of f as a complex function, replacing the variable t by z. Then f is analytic on the simply connected region $\mathbb{C}$ except for simple poles at i, $-i$ and 2. We define

   $$I = \int_\Gamma f(z)\,dz,$$

   where $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ is the contour shown in Figure 3.8. As you can see, Γ_1 and Γ_3 are line segments on the real axis, and Γ_2 and Γ_4 are semicircles.

   We choose $0 < \varepsilon < 1$ and $r > 3$, so that Γ is a simple-closed contour containing the point i.

2. The only pole of f inside the contour is i. Since

   $$f(z) = \frac{z}{(z-i)(z+i)(z-2)},$$

   we can apply the Cover-up Rule to see that

   $$\operatorname{Res}(f,i) = \frac{i}{2i(i-2)} = \frac{1}{2}\left(\frac{-i-2}{5}\right) = -\frac{1}{5} - \frac{i}{10}.$$

   It then follows from the Residue Theorem that

   $$I = 2\pi i \operatorname{Res}(f,i) = 2\pi i \times \left(-\frac{1}{5} - \frac{i}{10}\right) = \frac{\pi}{5} - \frac{2\pi i}{5}.$$

   Notice that this value of I is independent of both ε and r, provided that $\varepsilon < 1$ and $r > 3$ (so it does not change when we take limits later on).

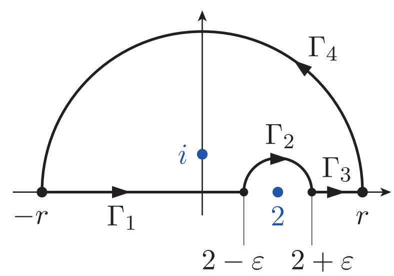


Figure 3.8 The contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$

3. Next we split up the integral into its constituent parts, obtaining

$$\begin{aligned} I &= \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_3} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \int_{\Gamma_4} f(z)\,dz \\ &= \int_{-r}^{2-\varepsilon} f(t)\,dt + \int_{2+\varepsilon}^{r} f(t)\,dt + \int_{\Gamma_2} f(z)\,dz + \int_{\Gamma_4} f(z)\,dz, \qquad (3.3) \end{aligned}$$

where, in the second line, we have written the first two integrals as real integrals.

4. Let us now apply the Estimation Theorem to obtain an upper estimate for the Γ_4 integral. Notice that $|z| = r$, for $z \in \Gamma_4$. Using the backwards form of the Triangle Inequality (applied twice), we see that for $z \in \Gamma_4$,

$$|(z^2+1)(z-2)| \geq (|z^2|-1)(|z|-2) = (r^2-1)(r-2).$$

Hence, for $z \in \Gamma_4$,

$$|f(z)| = \left|\frac{z}{(z^2+1)(z-2)}\right| \leq \frac{r}{(r^2-1)(r-2)}.$$

The length of Γ_4 is πr, so, because f is continuous on Γ_4, we can apply the Estimation Theorem to give

$$\left|\int_{\Gamma_4} f(z)\,dz\right| \leq \frac{r}{(r^2-1)(r-2)} \times \pi r = \frac{\pi r^2}{(r^2-1)(r-2)}, \qquad (3.4)$$

for $r > 3$.

5. To finish, we take limits of each expression in equation (3.3), first as $\varepsilon \to 0$ and then as $r \to \infty$. We can calculate

$$\lim_{\varepsilon\to 0} \int_{\Gamma_2} f(z)\,dz$$

using the Round-the-Pole Lemma. To use that lemma, we need the residue of f at 2, which can be obtained by the Cover-up Rule. It is

$$\operatorname{Res}(f,2) = \frac{2}{2^2+1} = \frac{2}{5}.$$

Now, observe that the semicircular path Γ_2 is traversed from $2-\varepsilon$ to $2+\varepsilon$ (not the other way round), so we must apply the Round-the-Pole Lemma to the reverse contour $\widetilde{\Gamma}_2$. We obtain

$$\lim_{\varepsilon\to 0} \int_{\Gamma_2} f(z)\,dz = -\lim_{\varepsilon\to 0} \int_{\widetilde{\Gamma}_2} f(z)\,dz = -\pi i \times \frac{2}{5} = -\frac{2\pi i}{5}.$$

Next we take the limit as $\varepsilon \to 0$ of each expression in equation (3.3). Then we see that the limit

$$\int_{-r}^{r} f(t)\,dt = \lim_{\varepsilon\to 0}\left(\int_{-r}^{2-\varepsilon} f(t)\,dt + \int_{2+\varepsilon}^{r} f(t)\,dt\right),$$

exists, and

$$I = \frac{\pi}{5} - \frac{2\pi i}{5} = \int_{-r}^{r} f(t)\,dt - \frac{2\pi i}{5} + \int_{\Gamma_4} f(z)\,dz,$$

using the value for I found in step 2. The Γ_4 integral tends to 0 as $r \to \infty$, by equation (3.4) and the corollary to Theorem 3.1. Hence

$$\int_{-\infty}^{\infty} f(t)\,dt = \lim_{r\to\infty} \int_{-r}^{r} f(t)\,dt = \frac{\pi}{5}.$$

Use the method from Example 3.5 in the following exercise.

Exercise 3.7

Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{t}{t^3 - 1}\,dt.$$

So far in this subsection we have been integrating rational functions. The integrand in the next example, however, is the product of a rational function and a trigonometric function. We can apply the Residue Theorem using a similar strategy to before, but with some important adjustments.

Example 3.6

Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2 + 4}\,dt.$$

Solution

1. Motivated by the earlier examples, we might try integrating the expression $(\cos z)/(z^2 + 4)$ over a suitable contour. However, if we do so, then we will encounter difficulties in step 5 when we take limits. Instead we work with the complex function

   $$f(z) = \frac{e^{iz}}{z^2 + 4} = \frac{e^{iz}}{(z - 2i)(z + 2i)}.$$

   Notice that if t is a real variable, then

   $$\frac{e^{it}}{t^2 + 4} = \frac{\cos t + i \sin t}{t^2 + 4} = \frac{\cos t}{t^2 + 4} + i \frac{\sin t}{t^2 + 4}. \tag{3.5}$$

   The function f has no poles on the real axis, so we use the contour integral

   $$I = \int_\Gamma f(z)\,dz,$$

   where $\Gamma = \Gamma_1 + \Gamma_2$ is the contour shown in Figure 3.9.

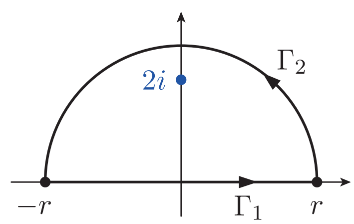


Figure 3.9 The contour $\Gamma = \Gamma_1 + \Gamma_2$

The function f is analytic on the simply connected region $\mathbb{C}$ except for simple poles at $2i$ and $-2i$. We choose $r > 2$, so that the pole $2i$ lies inside Γ (and $-2i$ lies outside Γ).

2. The residue of f at $2i$ can be found using the Cover-up Rule; it is

$$\operatorname{Res}(f, 2i) = \frac{e^{i\times 2i}}{2i+2i} = \frac{1}{4ie^2}.$$

Hence, by the Residue Theorem,

$$I = 2\pi i \times \frac{1}{4ie^2} = \frac{\pi}{2e^2}.$$

3. Next we split up the contour integral to give

$$\begin{aligned} I &= \int_{-r}^{r} \frac{e^{it}}{t^2+4}\,dt + \int_{\Gamma_2} \frac{e^{iz}}{z^2+4}\,dz \\ &= \int_{-r}^{r} \frac{\cos t}{t^2+4}\,dt + i\int_{-r}^{r} \frac{\sin t}{t^2+4}\,dt + \int_{\Gamma_2} \frac{e^{iz}}{z^2+4}\,dz. \end{aligned} \tag{3.6}$$

In the second line we have split the integral over t into its real and imaginary parts using equation (3.5).

4. Let us now obtain an upper estimate for the modulus of the Γ_2 integral. If $z \in \Gamma_2$, then $|z| = r$, so

$$\left|\frac{1}{z^2+4}\right| \le \frac{1}{|z^2|-4} = \frac{1}{r^2-4},$$

by the backwards form of the Triangle Inequality (since $r > 2$). Also, writing $z = x + iy$, we see that for $z \in \Gamma_2$,

$$|e^{iz}| = |e^{ix}e^{-y}| = |e^{ix}||e^{-y}| = 1 \times e^{-y} = e^{-y} \le 1,$$

because $y \ge 0$ on Γ_2. Therefore

$$|f(z)| = \left|\frac{e^{iz}}{z^2+4}\right| \le \frac{1}{r^2-4},$$

for $z \in \Gamma_2$. Since f is continuous on Γ_2, and Γ_2 has length πr, we can apply the Estimation Theorem to give

$$\left|\int_{\Gamma_2} \frac{e^{iz}}{z^2+4}\,dz\right| \le \frac{1}{r^2-4} \times \pi r = \frac{\pi r}{r^2-4}.$$

5. Now take the limit as $r \to \infty$ of each expression in equation (3.6). The estimate above shows that the Γ_2 integral tends to 0 as $r \to \infty$. Since $I = \pi/(2e^2)$, we see from equation (3.6) that

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+4}\,dt + i\int_{-\infty}^{\infty} \frac{\sin t}{t^2+4}\,dt = \frac{\pi}{2e^2}.$$

On equating real parts of this equation, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+4}\,dt = \frac{\pi}{2e^2}.$$

Notice also that, on equating imaginary parts, we obtain

$$\int_{-\infty}^{\infty} \frac{\sin t}{t^2+4}\,dt = 0,$$

which we would expect anyway, because the integrand is an odd function.

Now try the next exercise, in which there is a trigonometric function in the integrand *and* the integrand has a pole on the real axis.

Exercise 3.8

Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\sin 2t}{t^3+1}\,dt.$$

(*Hint*: You will need to use the Round-the-Pole Lemma.)

3.3 Theorems for integrating rational functions

In the previous subsection we used the Residue Theorem to evaluate several improper integrals involving rational functions. This subsection gives two general results which contain formulas for evaluating whole classes of such integrals under appropriate conditions.

Before we give these general results, however, let us prove the Round-the-Pole Lemma, illustrated in Figure 3.10, which we applied earlier in Example 3.5.

Lemma 3.1 Round-the-Pole Lemma

Suppose that f is a function that is analytic on a punctured disc $\{z : 0 < |z-\alpha| < \delta\}$ and has a simple pole at α. Let Γ be the upper half of the circle centred at α of radius ε, where $\varepsilon < \delta$, traversed from $\alpha+\varepsilon$ to $\alpha-\varepsilon$. Then

$$\lim_{\varepsilon\to 0} \int_\Gamma f(z)\,dz = \pi i \operatorname{Res}(f,\alpha).$$

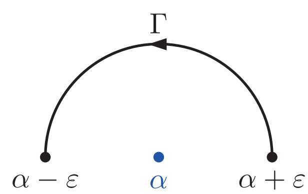


Figure 3.10 Semicircular contour

Proof Since f has a simple pole at α, its Laurent series about α is

$$f(z) = \frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha) + \cdots = \frac{\operatorname{Res}(f,\alpha)}{z-\alpha} + g(z),$$

where $a_{-1} = \operatorname{Res}(f,\alpha) \neq 0$ and the function $g(z) = a_0 + a_1(z-\alpha) + \cdots$ is analytic on $D = \{z : |z-\alpha| < \delta\}$. Thus, for $\varepsilon < \delta$, we have

$$\int_\Gamma f(z)\,dz = \operatorname{Res}(f,\alpha)\int_\Gamma \frac{1}{z-\alpha}\,dz + \int_\Gamma g(z)\,dz.$$

Since g is analytic on D, there is a number M such that

$$|g(z)| \le M, \quad \text{for } |z-\alpha| \le \tfrac{1}{2}\delta$$

(by the Boundedness Theorem – Theorem 5.3 of Unit A3). So, by the Estimation Theorem with $L = \pi\varepsilon$,

$$\left|\int_\Gamma g(z)\,dz\right| \le M \times \pi\varepsilon, \quad \text{for } 0 < \varepsilon \le \tfrac{1}{2}\delta.$$

Hence

$$\lim_{\varepsilon\to 0}\int_\Gamma g(z)\,dz = 0.$$

Now, using the standard parametrisation $\gamma(t) = \alpha + \varepsilon e^{it}$ $(t \in [0,\pi])$ for Γ, we have

$$\int_\Gamma \frac{1}{z-\alpha}\,dz = \int_0^\pi \frac{\varepsilon i e^{it}}{\varepsilon e^{it}}\,dt = \int_0^\pi i\,dt = \pi i,$$

so

$$\lim_{\varepsilon\to 0}\int_\Gamma f(z)\,dz = \pi i \operatorname{Res}(f,\alpha),$$

as required. ■

The next theorem can be used to integrate rational functions. It can be proved using the five-step strategy adopted in the previous subsection, following a similar procedure to Example 3.5. We omit the details.

Theorem 3.3

Let p and q be polynomial functions such that

- the degree of q exceeds that of p by at least two
- any poles of p/q on the real axis are simple.

Then

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\,dt = 2\pi i S + \pi i T,$$

where S is the sum of the residues of the function p/q at the poles in the upper half-plane, and T is the sum of the residues of p/q at the poles on the real axis.

For instance, to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{t}{(t^2+1)(t-2)}\,dt$$

considered in Example 3.5, we can apply Theorem 3.3 with $p(z) = z$ and $q(z) = (z^2+1)(z-2)$, because the degree of q exceeds that of p by $3-1=2$, and the only pole of p/q on the real axis is the simple pole at 2.

The other poles of p/q are at i and $-i$, and only i lies in the upper half-plane.

Now, we saw in Example 3.5 that

$$S = \operatorname{Res}(p/q, i) = -\frac{1}{5} - \frac{i}{10}$$

and

$$T = \operatorname{Res}(p/q, 2) = \frac{2}{5}.$$

Hence, by Theorem 3.3,

$$\int_{-\infty}^{\infty} \frac{t}{(t^2+1)(t-2)}\,dt = 2\pi i \times \left(-\frac{1}{5} - \frac{i}{10}\right) + \pi i \times \frac{2}{5} = \frac{\pi}{5}.$$

Exercise 3.9

Use Theorem 3.3 to evaluate

$$\int_{-\infty}^{\infty} \frac{1}{t(t-1)(t^2+1)}\,dt.$$

Exercise 3.10

Use Theorem 3.3 to show that if $a, b > 0$ and $a \neq b$, then

$$\int_{-\infty}^{\infty} \frac{1}{(t^2+a^2)(t^2+b^2)}\,dt = \frac{\pi}{ab(a+b)}.$$

Next we look at a theorem that can be used to evaluate integrals such as the one from Example 3.6.

Theorem 3.4

Let p and q be polynomial functions such that

- the degree of q exceeds that of p by at least one
- any poles of p/q on the real axis are simple.

Then, if $k > 0$,

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} e^{ikt}\, dt = 2\pi i S + \pi i T,$$

where S is the sum of the residues of the function

$$f(z) = \frac{p(z)}{q(z)} e^{ikz}$$

at the poles in the upper half-plane, and T is the sum of the residues of f at the poles on the real axis.

If p and q are real polynomial functions, then we can equate the real and imaginary parts of the equation

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} e^{ikt}\, dt = 2\pi i S + \pi i T$$

to obtain the values of the real improper integrals

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \cos kt\, dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \sin kt\, dt.$$

The following exercise will give you some practice at this technique.

Exercise 3.11

Use Theorem 3.4 to show that if $k > 0$, then

$$\int_{-\infty}^{\infty} \frac{e^{ikt}}{t}\, dt = i\pi.$$

Hence determine

$$\int_{-\infty}^{\infty} \frac{\cos kt}{t}\, dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin kt}{t}\, dt.$$

Notice that Theorem 3.3 requires the degree of q to exceed that of p by at least two, whereas Theorem 3.4 only requires the degree of q to exceed that of p by at least one. This difference arises because of the appearance of the exponential term e^{ikt} in Theorem 3.4. The key to the reduction from ‘two’ to ‘one’ lies in the following result, which is named after the French mathematician Camille Jordan, who proved the Jordan Curve Theorem, which you met in Unit B2.

Lemma 3.2 Jordan's Lemma

Let Γ be the upper half of the circle centred at 0 of radius r, traversed from r to $-r$, and suppose that f is a function that is continuous on Γ and satisfies

$$|f(z)| \leq M, \quad \text{for } z \in \Gamma.$$

Then, for $k > 0$, we have

$$\left| \int_{\Gamma} f(z)e^{ikz}\,dz \right| \leq \frac{M\pi}{k}.$$

Theorem 3.4 can be proved by a similar strategy to that of Example 3.6, using Jordan's Lemma to obtain an upper estimate for the integral of $(p(z)/q(z))e^{ikz}$ along a large semicircular contour. We omit the details of this proof and the proof of Jordan's Lemma.

Further exercises

Exercise 3.12

Use Theorems 3.3 and 3.4 to evaluate each of the following improper integrals.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{t^2}{(t^2+4)^2}\,dt$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos t}{t^2+9}\,dt$

Exercise 3.13

Evaluate each of the following improper integrals.

(a) $\displaystyle\int_{0}^{\infty} \frac{1}{(t+1)^2}\,dt$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{t}{t^4+1}\,dt$ (c) $\displaystyle\int_{0}^{\infty} \frac{t^2}{(t^2+4)^2}\,dt$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{t}{t^3+1}\,dt$ (e) $\displaystyle\int_{-\infty}^{\infty} \frac{t\sin \pi t}{1-t^2}\,dt$

Fourier transforms

The **Fourier transform** of a function $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ is the function

$$\widetilde{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt}\,dx,$$

provided that f is sufficiently well behaved that this improper integral exists.

Fourier transforms are extremely important tools in engineering and physics, where they are used to solve differential equations that model a range of physical phenomena. They are named after the French mathematician and physicist Joseph Fourier, whom you met in Unit B4 in a discussion of his work on Fourier series.

Note that some texts use other definitions of the Fourier transform of f such as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ixt}\,dx,$$

and some use the notation $\widehat{f}$ instead of $\widetilde{f}$.

Under suitable conditions, the function f can be recovered from $\widetilde{f}$ using the formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(t)e^{ixt}\,dt,$$

which is known as the **inverse Fourier transform**. These formulas involve improper integrals, and many such integrals can be evaluated using the techniques covered in this section.

For example, consider the function $f(x) = e^{-|x|}$, which has Fourier transform

$$\widetilde{f}(t) = \int_{-\infty}^{\infty} e^{-|x|}e^{-ixt}\,dx = \frac{2}{t^2+1},$$

as you can check by evaluating the integrals between $-\infty$ and 0, and between 0 and ∞, separately, and adding the two answers. Substituting $\widetilde{f}(t) = 2/(t^2+1)$ into the inverse Fourier transform formula, we obtain the integral

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2e^{ixt}}{t^2+1}\,dt.$$

Let us check that this integral really is equal to $f(x)$.

When $x > 0$, the integral can be evaluated using Theorem 3.4 with $p(t) = 2$ and $q(t) = t^2+1$. Let $g(t) = (p(t)/q(t))e^{ixt}$. Then Theorem 3.4 tells us that

$$I = \frac{1}{2\pi} \times 2\pi i \operatorname{Res}(g, i) = i \times \frac{2e^{ix\times i}}{2i} = e^{-x}.$$

By carrying out a similar calculation for $x < 0$ (using the substitution $s = -t$ to get the integral into a suitable form to apply Theorem 3.4), and evaluating a standard real integral in the special case $x = 0$, we can see that $I = e^{-|x|}$ for all values of x. Thus $I = f(x)$, so we have verified that the inverse Fourier transform does indeed reverse the action of the Fourier transform, in this special case.

4 Summing series

After working through this section, you should be able to:

- use the Residue Theorem to sum certain series of the forms

$$\sum_{n=1}^{\infty} h(n) \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n h(n),$$

  where h is an appropriate even function.

4.1 Sums of even functions

In this subsection we will use the theory of residues to sum certain real infinite series of the form

$$\sum_{n=1}^{\infty} h(n),$$

where h is an even function (that is, $h(z) = h(-z)$ for all z in the domain of h). In particular, we will evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \cdots,$$

which correspond to the even functions $h(z) = 1/z^2$ and $h(z) = 1/z^4$.

At first sight it may seem rather strange that a method for evaluating complex integrals can also be useful for summing real series. However, since we can integrate a function f along a simple-closed contour Γ by calculating

$$2\pi i \times (\text{the sum of the residues of } f \text{ at singularities inside } \Gamma),$$

it seems at least possible that we can represent a series as a sum of residues of a function. We may then be able to find the sum of the series by evaluating suitable contour integrals.

Let us describe this process in more detail. Recall first that the singularities of the function

$$g(z) = \pi \cot \pi z = \frac{\pi \cos \pi z}{\sin \pi z}$$

are simple poles at the points where $\sin \pi z = 0$, that is, at the integers $z = 0, \pm 1, \pm 2, \ldots$. Furthermore, the residue of g at each of these simple poles is 1 (see Example 1.5(a)).

To see why this is relevant, consider the function

$$f(z) = (\pi \cot \pi z)h(z),$$

where h is an even function that is analytic on $\mathbb{C}$ apart from a finite number of poles, none of which is an integer, except (possibly) 0. The singularities of f consist of the integers and the singularities of h.

The residue of f at a non-zero integer n is

$$\begin{aligned}\lim_{z\to n}(z-n)f(z) &= \lim_{z\to n}((z-n)\pi\cot\pi z)\lim_{z\to n}h(z)\\ &= \operatorname{Res}(g,n)\times h(n)\\ &= 1\times h(n) = h(n).\end{aligned}$$

Thus the sum of the residues of f at the *positive* integers is

$$h(1)+h(2)+\cdots = \sum_{n=1}^{\infty} h(n).$$

Note also that the sum of the residues of f at the *negative* integers is

$$h(-1)+h(-2)+\cdots = \sum_{n=1}^{\infty} h(-n) = \sum_{n=1}^{\infty} h(n),$$

since h is an even function.

The method we use is to integrate the function $f(z) = (\pi \cot \pi z)h(z)$ around the square contour S_N with vertices at the points

$$\left(N+\tfrac{1}{2}\right)(1+i),\ \left(N+\tfrac{1}{2}\right)(-1+i),\ \left(N+\tfrac{1}{2}\right)(-1-i),\ \left(N+\tfrac{1}{2}\right)(1-i),$$

where N is a positive integer large enough for S_N to contain all the non-zero poles $\alpha_1, \alpha_2, \ldots, \alpha_k$ of h (see Figure 4.1).

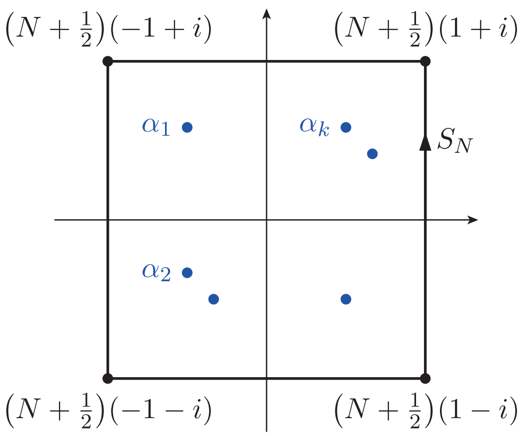


Figure 4.1 Square contour S_N containing the non-zero poles $\alpha_1, \alpha_2, \ldots, \alpha_k$ of h

We now apply the Residue Theorem to the function

$$f(z) = (\pi \cot \pi z)h(z)$$

and the contour S_N on the simply connected region $\{z : |z| < 3N\}$, which

contains S_N and contains only finitely many singularities of f. We obtain

$$\begin{aligned}
&\int_{S_N} f(z)\,dz \\
&= \int_{S_N} (\pi\cot\pi z)h(z)\,dz \\
&= 2\pi i\left(\sum_{n=1}^{N}\operatorname{Res}(f,n) + \sum_{n=1}^{N}\operatorname{Res}(f,-n) + \operatorname{Res}(f,0) + \sum_{j=1}^{k}\operatorname{Res}(f,\alpha_j)\right) \\
&= 2\pi i\left(2\sum_{n=1}^{N} h(n) + \operatorname{Res}(f,0) + \sum_{j=1}^{k}\operatorname{Res}(f,\alpha_j)\right),
\end{aligned}$$

since $\operatorname{Res}(f,n) = h(n)$ and $\operatorname{Res}(f,-n) = h(-n) = h(n)$.

We now let N tend to ∞. If the function h has been chosen so that

$$\lim_{N\to\infty}\int_{S_N} (\pi\cot\pi z)h(z)\,dz = 0,$$

then we obtain

$$0 = 2\pi i\left(2\sum_{n=1}^{\infty} h(n) + \operatorname{Res}(f,0) + \sum_{j=1}^{k}\operatorname{Res}(f,\alpha_j)\right).$$

Rearranging this gives

$$\sum_{n=1}^{\infty} h(n) = -\frac{1}{2}\left(\operatorname{Res}(f,0) + \sum_{j=1}^{k}\operatorname{Res}(f,\alpha_j)\right).$$

Thus we have obtained the sum of the series in terms of the residues of f at 0 and at the non-zero poles of h.

We summarise this result in the following theorem.

Theorem 4.1

Let h be an even function that is analytic on $\mathbb{C}$ except for poles at the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ (none of which is an integer), and possibly at 0, and let S_N be the square contour with vertices at $\left(N+\frac{1}{2}\right)(\pm 1 \pm i)$. Suppose also that the function $f(z) = (\pi\cot\pi z)h(z)$ is such that

$$\lim_{N\to\infty}\int_{S_N} f(z)\,dz = 0. \tag{4.1}$$

Then

$$\sum_{n=1}^{\infty} h(n) = -\frac{1}{2}\left(\operatorname{Res}(f,0) + \sum_{j=1}^{k}\operatorname{Res}(f,\alpha_j)\right).$$

Condition (4.1) is not nearly as artificial as it may seem, in view of the following result (which we prove at the end of the next subsection).

Lemma 4.1

For each $N = 1, 2, \ldots,$

$$|\cot \pi z| \le 2, \quad \text{for } z \in S_N,$$

where S_N is the square contour with vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$.

Using this result, we can prove that condition (4.1) holds for a wide variety of functions h. The reason why we chose a square contour S_N instead of (say) a circular one is that the proof of Lemma 4.1 is simpler for square contours.

To apply Theorem 4.1 to sum a particular series, it is useful to know the Laurent series about 0 for cot. This series can be found by taking the reciprocal of the Taylor series about 0 for tan, which we found in Subsection 4.2 of Unit B3 to be

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots, \quad \text{for } |z| < \pi/2.$$

For $0 < |z| < \pi/2$, we see that

$$\cot z = (\tan z)^{-1} = z^{-1}\left(1 + \left(\frac{1}{3}z^2 + \frac{2}{15}z^4 + \cdots\right)\right)^{-1}.$$

We can now apply the Composition Rule for power series (Theorem 4.3 of Unit B3) to the functions

$$f(z) = \frac{1}{3}z^2 + \frac{2}{15}z^4 + \cdots$$

and

$$g(w) = (1 + w)^{-1} = 1 - w + w^2 - \cdots.$$

Both f and g are expressed as Taylor series about 0, and $f(0) = 0$, so we see that

$$\begin{aligned}\cot z &= z^{-1}\left(1 - \left(\frac{1}{3}z^2 + \frac{2}{15}z^4 + \cdots\right) + \left(\frac{1}{3}z^2 + \cdots\right)^2 - \cdots\right)\\ &= z^{-1}\left(1 - \frac{1}{3}z^2 - \frac{1}{45}z^4 - \cdots\right),\end{aligned}$$

for $0 < |z| < r$, where r is some positive number. Hence

$$\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \cdots, \quad \text{for } 0 < |z| < r. \tag{4.2}$$

Observe that only odd powers of z appear in this series, since cot is an odd function.

Example 4.1

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution

The function $h(z) = 1/z^2$ is an even function and it is analytic on $\mathbb{C}$ apart from a pole of order two at 0.

The function

$$f(z) = (\pi \cot \pi z)/z^2$$

has a pole of order three at 0. We could find the residue of f at 0 by applying Theorem 1.3 with $k = 3$; however, it is simpler to use the Laurent series about 0 for cot given in equation (4.2). We see that

$$\begin{aligned}\frac{\pi \cot \pi z}{z^2} &= \frac{\pi}{z^2}\left(\frac{1}{\pi z} - \frac{1}{3}\pi z - \cdots\right)\\ &= \frac{1}{z^3} - \frac{\pi^2}{3z} - \cdots,\end{aligned}$$

so $\operatorname{Res}(f, 0) = -\dfrac{\pi^2}{3}$.

We now check condition (4.1). If z lies on the contour S_N, then

$$|z| \geq N + \tfrac{1}{2},$$

so, by Lemma 4.1,

$$|f(z)| = \left|\frac{\pi \cot \pi z}{z^2}\right| \leq \frac{2\pi}{\left(N + \frac{1}{2}\right)^2}, \quad \text{for } z \in S_N.$$

Hence, by the Estimation Theorem with $M = 2\pi/\left(N + \frac{1}{2}\right)^2$ and $L = 4(2N + 1)$, we see that

$$\left|\int_{S_N} f(z)\, dz\right| \leq \frac{2\pi}{\left(N + \frac{1}{2}\right)^2} \times 4(2N + 1) = \frac{32\pi}{2N + 1},$$

which tends to 0 as $N \to \infty$. Thus condition (4.1) holds.

It follows from Theorem 4.1 that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2}\operatorname{Res}(f, 0) = -\frac{1}{2}\left(-\frac{\pi^2}{3}\right) = \frac{\pi^2}{6}.$$

Exercise 4.1

Determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Remark

We can use the method of Example 4.1 to sum the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k},$$

whenever k is a positive *even* integer, because in this case $h(z) = 1/z^k$ is an even function.

However, when k is *odd*, the method fails, and much remains unknown about these sums. For example, it is not known in general whether the values of the sums for k odd are irrational, although for the case $k = 3$ this was proved in 1979 by the Greek–French mathematician Roger Apéry (1916–1994). To pay tribute to this achievement, his tombstone in Paris bears the inscription

$$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots \neq \frac{p}{q}.$$

We will return to the study of sums of this form when we consider the zeta function in Section 5 of Unit C2.

In the argument leading to the statement of Theorem 4.1, we saw that the residue of the function

$$f(z) = (\pi \cot \pi z)h(z)$$

at a non-zero integer n is given by

$$\operatorname{Res}(f, n) = h(n).$$

In fact, this result is also valid for $n = 0$, provided that h is analytic at 0. This provides a quick way of evaluating $\operatorname{Res}(f, 0)$.

Observation

If $f(z) = (\pi \cot \pi z)h(z)$, where h is analytic at 0, then

$$\operatorname{Res}(f, 0) = h(0).$$

You can use this result in the following exercise.

Exercise 4.2

Determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

(You calculated the relevant residues in Exercise 1.5(b).)

4.2 Alternating sums of even functions

In this subsection we describe a method similar to that used in the preceding subsection to sum series of the form

$$\sum_{n=1}^{\infty} (-1)^n h(n),$$

where h is an even function. Examples of such series are

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \cdots$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = -\frac{1}{3} + \frac{1}{15} - \frac{1}{35} + \frac{1}{63} - \cdots.$$

This time we replace the function $g(z) = \pi \cot \pi z$ with the function

$$g(z) = \pi \operatorname{cosec} \pi z = \frac{\pi}{\sin \pi z}.$$

Once again, the singularities of this function are simple poles at the integers, but this time the residue at the integer n is $(-1)^n$, by Example 1.5(b).

To see the effect of this change, consider the function

$$f(z) = (\pi \operatorname{cosec} \pi z) h(z),$$

where h is an even function that is analytic on $\mathbb{C}$ apart from a finite number of poles, none of which is an integer, except (possibly) 0. Then the residue of f at a non-zero integer n is

$$\begin{aligned}\lim_{z \to n} (z - n) f(z) &= \lim_{z \to n} ((z - n)\pi \operatorname{cosec} \pi z) \lim_{z \to n} h(z) \\ &= \operatorname{Res}(g, n) \times h(n) \\ &= (-1)^n h(n).\end{aligned}$$

As before, this result extends to $n = 0$.

Observation

If $f(z) = (\pi \operatorname{cosec} \pi z) h(z)$, where h is analytic at 0, then

$$\operatorname{Res}(f, 0) = h(0).$$

We now integrate the function $f(z) = (\pi \operatorname{cosec} \pi z)h(z)$ around the same square contour S_N, where N is a positive integer large enough for S_N to contain all the non-zero poles $\alpha_1, \alpha_2, \ldots, \alpha_k$ of h (see Figure 4.1). It follows from the Residue Theorem, applied to the function $f(z) = (\pi \operatorname{cosec} \pi z)h(z)$ and the contour S_N on the simply connected region $\{z : |z| < 3N\}$, that

$$\begin{aligned}
&\int_{S_N} f(z)\,dz \\
&= \int_{S_N} (\pi \operatorname{cosec} \pi z)h(z)\,dz \\
&= 2\pi i\left(\sum_{n=1}^{N} \operatorname{Res}(f,n) + \sum_{n=1}^{N} \operatorname{Res}(f,-n) + \operatorname{Res}(f,0) + \sum_{j=1}^{k} \operatorname{Res}(f,\alpha_j)\right) \\
&= 2\pi i\left(2\sum_{n=1}^{N}(-1)^n h(n) + \operatorname{Res}(f,0) + \sum_{j=1}^{k} \operatorname{Res}(f,\alpha_j)\right),
\end{aligned}$$

since $\operatorname{Res}(f,n) = (-1)^n h(n)$ and $\operatorname{Res}(f,-n) = (-1)^{-n}h(-n) = (-1)^n h(n)$, because h is an even function.

We now let N tend to ∞. If h has been chosen so that

$$\lim_{N\to\infty} \int_{S_N} (\pi \operatorname{cosec} \pi z)h(z)\,dz = 0,$$

then we obtain

$$0 = 2\pi i\left(2\sum_{n=1}^{\infty}(-1)^n h(n) + \operatorname{Res}(f,0) + \sum_{j=1}^{k} \operatorname{Res}(f,\alpha_j)\right).$$

Rearranging this gives

$$\sum_{n=1}^{\infty}(-1)^n h(n) = -\frac{1}{2}\left(\operatorname{Res}(f,0) + \sum_{j=1}^{k} \operatorname{Res}(f,\alpha_j)\right).$$

Thus we have obtained the sum of the series in terms of the residues of f at 0 and at the non-zero poles of h.

We summarise this result in the following theorem.

Theorem 4.2

Let h be an even function that is analytic on $\mathbb{C}$ except for poles at the points $\alpha_1, \alpha_2, \ldots, \alpha_k$ (none of which is an integer), and possibly at 0, and let S_N be the square contour with vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$. Suppose also that the function $f(z) = (\pi \operatorname{cosec} \pi z)h(z)$ is such that

$$\lim_{N\to\infty} \int_{S_N} f(z)\,dz = 0. \tag{4.3}$$

Then

$$\sum_{n=1}^{\infty}(-1)^n h(n) = -\frac{1}{2}\left(\operatorname{Res}(f,0) + \sum_{j=1}^{k} \operatorname{Res}(f,\alpha_j)\right).$$

Condition (4.3) holds for a wide variety of functions h, and can be established in particular cases by using the following analogue of Lemma 4.1, proved at the end of this subsection.

Lemma 4.2

For each $N = 1, 2, \ldots,$

$$|\operatorname{cosec} \pi z| \le 1, \quad \text{for } z \in S_N,$$

where S_N is the square contour with vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$.

Let us now use Theorem 4.2 to sum a particular series.

Example 4.2

Determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

Solution

The function $h(z) = 1/(4z^2 - 1)$ is an even function and it is analytic on $\mathbb{C}$ apart from simple poles at $\frac{1}{2}$ and $-\frac{1}{2}$.

The residues of the function

$$f(z) = (\pi \operatorname{cosec} \pi z)/(4z^2 - 1)$$

at $\frac{1}{2}$ and $-\frac{1}{2}$ are both $\pi/4$ (see Exercise 1.5(a)).

Since h is analytic at 0,

$$\operatorname{Res}(f, 0) = h(0) = -1.$$

We now check condition (4.3). If z lies on the contour S_N, then

$$|z| \ge N + \tfrac{1}{2},$$

so we can apply the backwards form of the Triangle Inequality to give

$$|4z^2 - 1| \ge |4z^2| - 1 \ge 4\left(N + \tfrac{1}{2}\right)^2 - 1.$$

It then follows from Lemma 4.2 that

$$|f(z)| = \left|\frac{\pi \operatorname{cosec} \pi z}{4z^2 - 1}\right| \le \frac{\pi}{4\left(N + \frac{1}{2}\right)^2 - 1}, \quad \text{for } z \in S_N.$$

Hence, by the Estimation Theorem,

$$\left|\int_{S_N} f(z)\, dz\right| \le \frac{\pi}{4\left(N + \frac{1}{2}\right)^2 - 1} \times 4(2N + 1),$$

which tends to 0 as $N \to \infty$. Thus condition (4.3) holds.

It follows from Theorem 4.2 that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} &= -\frac{1}{2}\left(\operatorname{Res}(f,0)+\operatorname{Res}\left(f,\tfrac{1}{2}\right)+\operatorname{Res}\left(f,-\tfrac{1}{2}\right)\right)\\ &= -\frac{1}{2}\left(-1+\frac{\pi}{4}+\frac{\pi}{4}\right)\\ &= \frac{1}{2}-\frac{\pi}{4}.\end{aligned}$$

In the next exercise it will help you to know that the Laurent series about 0 for cosec is

$$\operatorname{cosec} z = \frac{1}{z}+\frac{1}{6}z+\frac{7}{360}z^3+\cdots.$$

This series can be obtained by taking the reciprocal of the Taylor series about 0 for sin, in a similar way to how we obtained the Laurent series about 0 for cot from the Taylor series about 0 for tan in Subsection 4.1.

Exercise 4.3

Determine the sum of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

We conclude this subsection by proving Lemmas 4.1 and 4.2.

Lemmas 4.1 and 4.2 (combined)

For each $N = 1, 2, \ldots,$

$$|\cot \pi z| \leq 2 \quad \text{and} \quad |\operatorname{cosec} \pi z| \leq 1, \quad \text{for } z \in S_N,$$

where S_N is the square contour with vertices at $\left(N+\frac{1}{2}\right)(\pm 1 \pm i)$.

Proof Let $z = x + iy$. Then

$$\begin{aligned}\sin \pi z &= \sin(\pi x + i\pi y)\\ &= \sin \pi x \cos i\pi y + \cos \pi x \sin i\pi y\\ &= \sin \pi x \cosh \pi y + i \cos \pi x \sinh \pi y.\end{aligned}$$

Therefore

$$|\sin \pi z|^2 = (\sin \pi x \cosh \pi y)^2 + (\cos \pi x \sinh \pi y)^2. \tag{4.4}$$

If $z = x + iy$ lies on one of the vertical sides of S_N, then $x = \pm\left(N+\frac{1}{2}\right)$, so $\sin \pi x = \pm 1$ and $\cos \pi x = 0$. Hence, from equation (4.4),

$$|\sin \pi z|^2 = \cosh^2 \pi y \geq 1.$$

If $z = x + iy$ lies on one of the horizontal sides of S_N, then $|y| \geq 1$, so

$$|\sinh \pi y| = \sinh \pi|y| \geq \pi|y| \geq 1,$$

since

$$\sinh t = t + \frac{1}{3!}t^3 + \cdots \geq t, \quad \text{for } t \geq 0.$$

Hence, using equation (4.4) and the inequality $\cosh^2 \pi y \geq \sinh^2 \pi y$, we see that

$$|\sin \pi z|^2 \geq (\sin \pi x \sinh \pi y)^2 + (\cos \pi x \sinh \pi y)^2 = \sinh^2 \pi y \geq 1.$$

We have now verified that $|\sin \pi z| \geq 1$ for any point $z \in S_N$. Therefore $|\operatorname{cosec} \pi z| \leq 1$. Furthermore, for $z \in S_N$, we have

$$|\cot \pi z|^2 = \left|\cot^2 \pi z\right| = \left|\operatorname{cosec}^2 \pi z - 1\right| \leq \left|\operatorname{cosec}^2 \pi z\right| + 1 \leq 2,$$

so $|\cot \pi z| \leq \sqrt{2} \leq 2$, as required. ■

Further exercises

Exercise 4.4

Determine the sum of each of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2+1}$

(You calculated the relevant residues in Exercise 1.5(c) and (d).)

Exercise 4.5

By taking $h(z) = 1/(z^2 - \alpha^2)$ in Theorem 4.1, obtain the formula

$$\pi \cot \pi\alpha = \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2}, \quad \text{for } \alpha \in \mathbb{C} - \mathbb{Z}.$$

5 Analytic continuation

After working through this section, you should be able to:

- define a *generalised argument function* and a *generalised logarithm function*, identify their associated cut planes, and evaluate such functions
- determine *direct analytic continuations* of certain analytic functions
- evaluate certain improper integrals involving logarithms or non-integer powers
- establish that two given functions are *indirect analytic continuations* of each other.

5.1 Generalised logarithm functions

In Unit A4 you learned that the principal logarithm function Log is analytic on the cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$. This subsection is about a more general family of logarithm functions, which will be used later in this section to discuss analytic continuation, and then used again in the next unit to represent winding numbers.

The first step towards defining this family of logarithm functions is to introduce a family of argument functions, which generalise the principal argument function Arg.

Definition

For $\phi \in \mathbb{R}$, the function Arg_ϕ is defined by

$$\mathrm{Arg}_\phi(z) = \theta \quad (z \in \mathbb{C} - \{0\}),$$

where θ is the argument of z lying in the interval $(\phi - 2\pi, \phi]$.

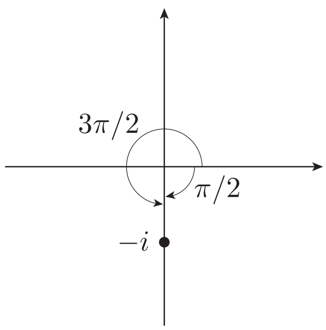


Figure 5.1 Two arguments of $-i$

For example,

$$\mathrm{Arg}_\pi(-i) = -\pi/2,$$

since $-\pi/2$ is the argument of $-i$ that lies in $(-\pi, \pi]$, whereas

$$\mathrm{Arg}_{2\pi}(-i) = 3\pi/2,$$

since $3\pi/2$ is the argument of $-i$ that lies in $(0, 2\pi]$ (see Figure 5.1).

Exercise 5.1

Evaluate each of the following expressions.

(a) $\mathrm{Arg}_\pi(i)$ (b) $\mathrm{Arg}_0(-1)$ (c) $\mathrm{Arg}_{3\pi/2}(1 - i)$

Since $-\pi < \mathrm{Arg}_\pi z \leq \pi$, it is evident that the function Arg_π is just the principal argument function Arg. Now, Arg is continuous when restricted to the cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ (by Example 2.3 of Unit A3), and in fact each of these generalised argument functions is continuous on an appropriate cut plane.

Definition

For $\phi \in \mathbb{R}$, the cut plane $\mathbb{C}_\phi$ is defined by

$$\mathbb{C}_\phi = \{re^{i\theta} : r > 0,\ \phi - 2\pi < \theta < \phi\}.$$

For example,

$$\mathbb{C}_\pi = \{re^{i\theta} : r > 0,\ -\pi < \theta < \pi\} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\},$$

illustrated in Figure 5.2(a), and

$$\mathbb{C}_{2\pi} = \{re^{i\theta} : r > 0,\ 0 < \theta < 2\pi\} = \mathbb{C} - \{x \in \mathbb{R} : x \geq 0\},$$

illustrated in Figure 5.2(b).

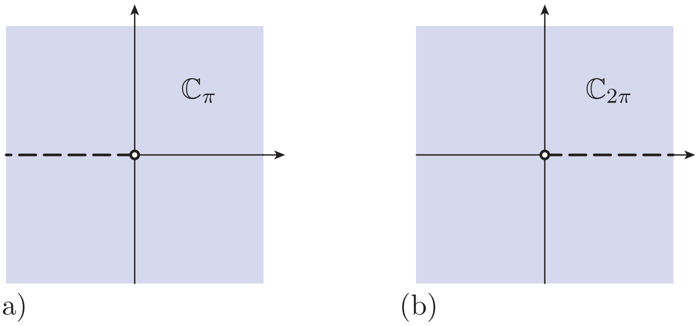


Figure 5.2 The cut planes (a) $\mathbb{C}_\pi$ and (b) $\mathbb{C}_{2\pi}$

Notice that

$$\mathbb{C}_{\phi+2n\pi} = \mathbb{C}_\phi, \quad \text{for } n \in \mathbb{Z},$$

since any two arguments of a complex number differ by an integer multiple of 2π. For example,

$$\cdots = \mathbb{C}_{-\pi} = \mathbb{C}_\pi = \mathbb{C}_{3\pi} = \mathbb{C}_{5\pi} = \cdots.$$

Exercise 5.2

Sketch each of the following cut planes.

(a) $\mathbb{C}_0$ (b) $\mathbb{C}_{3\pi/2}$

Theorem 5.1

For all $\phi \in \mathbb{R}$, Arg_ϕ is continuous on $\mathbb{C}_\phi$.

Proof By definition, $\operatorname{Arg}_\phi(z)$ is the argument of z that lies in $(\phi - 2\pi, \phi]$. Therefore $\operatorname{Arg}_\phi(z) - \phi + \pi$ lies in $(-\pi, \pi]$ and is an argument of

$$ze^{-i\phi}e^{i\pi} = -ze^{-i\phi}.$$

Hence

$$\operatorname{Arg}_\phi(z) - \phi + \pi = \operatorname{Arg}\big(-ze^{-i\phi}\big),$$

which gives

$$\operatorname{Arg}_\phi(z) = \phi - \pi + \operatorname{Arg}\big(-ze^{-i\phi}\big), \quad \text{for } z \in \mathbb{C} - \{0\}. \tag{5.1}$$

Observe that if $z \in \mathbb{C}_\phi$, then $-ze^{-i\phi} = ze^{-i\phi}e^{i\pi} \in \mathbb{C}_\pi$. Since the function $z \longmapsto -ze^{-i\phi}$ is continuous, and Arg is continuous on $\mathbb{C}_\pi$, we see from equation (5.1) that the restriction of Arg_ϕ to $\mathbb{C}_\phi$ can be expressed as a composition of continuous functions. Hence Arg_ϕ is continuous on $\mathbb{C}_\phi$. ■

We can use the functions Arg_ϕ to define a new family of logarithm functions.

Definition

For $\phi \in \mathbb{R}$, the function Log_ϕ is defined by

$$\operatorname{Log}_\phi(z) = \log|z| + i\operatorname{Arg}_\phi(z) \quad (z \in \mathbb{C} - \{0\}).$$

For example, since $\operatorname{Arg}_\pi = \operatorname{Arg}$, the function Log_π is just the principal logarithm function,

$$\operatorname{Log}_\pi = \operatorname{Log}.$$

Note that if $z \in \mathbb{C} - \{0\}$, then $\operatorname{Log}_\phi(z)$ *is* a logarithm of z, since

$$\begin{aligned}\exp(\operatorname{Log}_\phi(z)) &= \exp(\log|z| + i\operatorname{Arg}_\phi(z))\\ &= |z|e^{i\operatorname{Arg}_\phi(z)}\\ &= z,\end{aligned}$$

because $\operatorname{Arg}_\phi(z)$ is an argument of z.

Moreover, we can show that the function Log_ϕ is analytic on the cut plane $\mathbb{C}_\phi$ and that its derivative has the same rule as that of the principal logarithm function Log.

Theorem 5.2

For all $\phi \in \mathbb{R}$, the function Log_ϕ is analytic on $\mathbb{C}_\phi$ with derivative

$$\operatorname{Log}_\phi'(z) = \frac{1}{z}, \quad \text{for } z \in \mathbb{C}_\phi.$$

Proof For $z \in \mathbb{C}_\phi$ we have, by equation (5.1),

$$\begin{aligned}\operatorname{Log}_\phi(z) &= \log|z| + i\operatorname{Arg}_\phi(z)\\ &= \log|z| + i(\phi - \pi + \operatorname{Arg}(-ze^{-i\phi}))\\ &= \log|-ze^{-i\phi}| + i(\phi - \pi + \operatorname{Arg}(-ze^{-i\phi}))\\ &= i(\phi - \pi) + \operatorname{Log}(-ze^{-i\phi}).\end{aligned}$$

Now, the function $z \longmapsto -ze^{-i\phi}$ is an analytic function from $\mathbb{C}_\phi$ onto $\mathbb{C}_\pi$, and Log is analytic on $\mathbb{C}_\pi$, so we can apply the Chain Rule to see that Log_ϕ is analytic on $\mathbb{C}_\phi$ and

$$\operatorname{Log}_\phi'(z) = 0 + \frac{-e^{-i\phi}}{-ze^{-i\phi}} = \frac{1}{z},$$

for $z \in \mathbb{C}_\phi$, as required. ■

Exercise 5.3

Evaluate each of the following expressions.

(a) $\operatorname{Log}_{3\pi}(-i)$ (b) $\operatorname{Log}_{2\pi}(2)$ (c) $\operatorname{Log}_{3\pi/2}(-3)$

5.2 Direct analytic continuation

An analytic function f is usually specified by giving a rule $f(z)$ and a region $\mathcal{R}$ on which f is analytic. For example, the sum function f of the power series $1 + z + z^2 + \cdots$ is

$$f(z) = 1 + z + z^2 + \cdots \quad (|z| < 1), \tag{5.2}$$

and f is analytic on the open unit disc $D = \{z : |z| < 1\}$, the disc of convergence of this power series. Since this power series is a geometric series, we could equally well have defined the function f by using the formula for the sum of this series, that is,

$$f(z) = \frac{1}{1-z} \quad (|z| < 1).$$

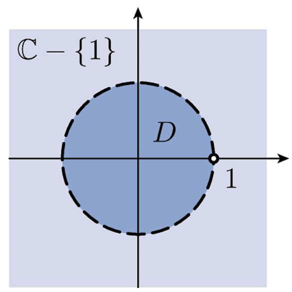


Figure 5.3 The open unit disc is contained in $\mathbb{C} - \{1\}$

But the expression $1/(1-z)$ is defined for all z in the larger region $\mathbb{C} - \{1\}$ (see Figure 5.3). Thus f is the restriction to D of the analytic function

$$g(z) = \frac{1}{1-z} \quad (z \in \mathbb{C} - \{1\}). \tag{5.3}$$

Put another way, g is an **analytic extension** of f from D to $\mathbb{C} - \{1\}$; that is, g is a function that is analytic on the larger set $\mathbb{C} - \{1\}$, and that agrees with f on the open unit disc D.

This notion of an analytic extension also arose in Unit B4, in connection with the idea of a removable singularity. For example, the function

$$f(z) = \frac{\sin z}{z} \quad (z \in \mathbb{C} - \{0\}) \tag{5.4}$$

is analytic and it has a removable singularity at 0 because the function

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \begin{cases} \dfrac{\sin z}{z}, & z \neq 0, \\ 1, & z = 0, \end{cases}$$

is analytic on $\mathbb{C}$ (see Subsection 1.3 of Unit B4). Thus g is an analytic extension of f from $\mathbb{C} - \{0\}$ to $\mathbb{C}$.

Example 5.1

Determine an analytic extension of the function

$$f(z) = \sum_{n=0}^{\infty} (-1)^{n+1}(z-2)^n \quad (|z-2| < 1).$$

Solution

Since

$$\sum_{n=0}^{\infty}(-1)^{n+1}(z-2)^n = -1 + (z-2) - (z-2)^2 + \cdots$$

is a geometric series with sum

$$\frac{-1}{1+(z-2)} = \frac{1}{1-z}, \quad \text{for } |z-2| < 1,$$

we deduce that the function

$$g(z) = \frac{1}{1-z} \quad (z \in \mathbb{C} - \{1\})$$

is an analytic extension of f to $\mathbb{C} - \{1\}$.

Exercise 5.4

Determine an analytic extension of each of the following functions.

(a) $f(z) = \sum_{n=0}^{\infty}(2z)^n \quad \left(|z| < \frac{1}{2}\right)$

(b) $f(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \quad (|z-1| < 1)$

For a given analytic function f, it is natural to seek the largest region to which f can be extended analytically. For example, the function f defined by equation (5.2) can be analytically extended to $\mathbb{C} - \{1\}$ by equation (5.3), but it cannot be extended any further because the function $g(z) = 1/(1-z)$ has a pole at 1. Similarly, the function f defined by equation (5.4) can be analytically extended to the whole of $\mathbb{C}$, and this is clearly the largest possible such region.

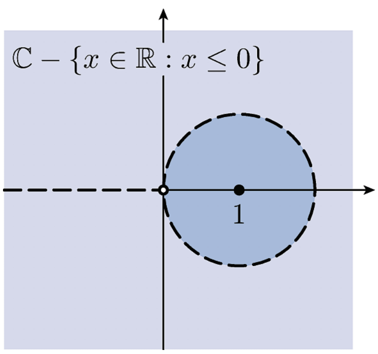


Figure 5.4 The open disc $\{z : |z-1| < 1\}$ is contained in $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$

For some functions f, however, there is no unique largest region in $\mathbb{C}$ to which f can be extended analytically. For example, in Exercise 5.4(b) you saw that the function

$$f(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \quad (|z-1| < 1)$$

has the analytic extension

$$g(z) = \operatorname{Log} z \quad (z \in \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\})$$

(see Figure 5.4).

Now, it is certainly not possible to extend this function g to a region that is larger than $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$, because the function Log is not analytic at any point of the negative real axis. So it might appear that we have

found the largest region to which f can be analytically extended. However, consider the function

$$h(z) = \mathrm{Log}_{3\pi/2}(z) \quad (z \in \mathbb{C}_{3\pi/2}).$$

Here $h(z) = \log|z| + i\,\mathrm{Arg}_{3\pi/2}(z)$ and $\mathbb{C}_{3\pi/2}$ is the cut plane illustrated in Figure 5.5.

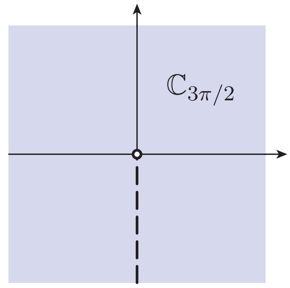


Figure 5.5 The cut plane $\mathbb{C}_{3\pi/2}$

Because

$$\mathrm{Arg}_{3\pi/2}(z) = \mathrm{Arg}\, z, \quad \text{for } \mathrm{Re}\, z > 0,$$

we have

$$h(z) = g(z), \quad \text{for } \mathrm{Re}\, z > 0,$$

so

$$h(z) = f(z), \quad \text{for } |z - 1| < 1.$$

Thus h is also an analytic extension of f, but it extends f to the region $\mathbb{C}_{3\pi/2}$ which is neither smaller nor larger than $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$. Thus it does not always make sense to seek *the* largest region to which a given analytic function can be extended. Instead we introduce a related idea called *analytic continuation*, and discuss to what extent a given analytic function can be analytically continued.

Definition

Let f and g be analytic functions whose domains are the regions $\mathcal{R}$ and $\mathcal{S}$, respectively. Then f and g are **direct analytic continuations** of each other if there is a region $\mathcal{T} \subseteq \mathcal{R} \cap \mathcal{S}$ such that

$$f(z) = g(z), \quad \text{for } z \in \mathcal{T}.$$

We also say that g is a **direct analytic continuation** of f from $\mathcal{R}$ to $\mathcal{S}$, and vice versa.

Two possible arrangements of the regions $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ are illustrated in Figure 5.6.

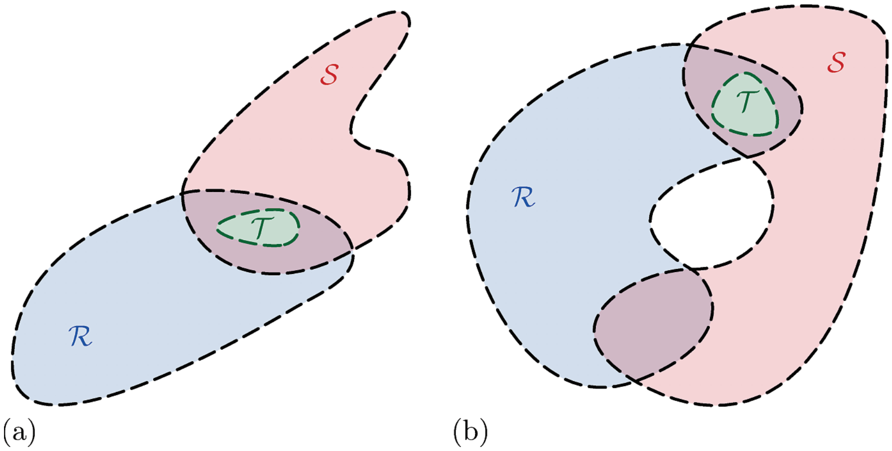


Figure 5.6 A region $\mathcal{T}$ inside the intersection of two other regions $\mathcal{R}$ and $\mathcal{S}$, where (a) $\mathcal{R} \cap \mathcal{S}$ is itself a region, (b) $\mathcal{R} \cap \mathcal{S}$ is not a region (it is disconnected)

Remarks

1. The word 'direct' appears in this definition because there is a notion of indirect analytic continuation, which we discuss in Subsection 5.4.
2. Notice that, by the Uniqueness Theorem (Theorem 5.5 of Unit B3), there can be at most one function g that is analytic on $\mathcal{S}$ and agrees with f on the region $\mathcal{T}$. This is often referred to as the *uniqueness of analytic continuation*.
3. Note that we do not insist in this definition that f and g are equal throughout $\mathcal{R} \cap \mathcal{S}$. However, if $\mathcal{R} \cap \mathcal{S}$ is a region (as in Figure 5.6(a)), then the equality of f and g on $\mathcal{T}$ implies their equality on $\mathcal{R} \cap \mathcal{S}$, by the Uniqueness Theorem.
4. The definitions above, which are symmetric with respect to f and g, apply in the special case for which $\mathcal{R} \subseteq \mathcal{S}$. Then only the direct analytic continuation of f from $\mathcal{R}$ to $\mathcal{S}$ is of interest.

Example 5.2

Prove that each of the following pairs of analytic functions f and g are direct analytic continuations of each other.

(a) $f(z) = \displaystyle\sum_{n=0}^{\infty} z^n \quad (|z| < 1), \quad g(z) = \frac{1}{1-z} \quad (z \in \mathbb{C} - \{1\})$

(b) $f(z) = \operatorname{Log}_{\pi}(z) \quad (z \in \mathbb{C}_{\pi}), \quad g(z) = \operatorname{Log}_{3\pi/2}(z) \quad (z \in \mathbb{C}_{3\pi/2})$

Solution

(a) Here $\mathcal{R} = \{z : |z| < 1\}$ and $\mathcal{S} = \mathbb{C} - \{1\}$. Since f and g agree on the region $\mathcal{R} = \{z : |z| < 1\} \subseteq \mathcal{S}$, as you saw at the start of this subsection, we deduce that g is a direct analytic continuation of f from $\mathcal{R}$ to $\mathcal{S}$.

(b) Here $\mathcal{R} = \mathbb{C}_{\pi}$ and $\mathcal{S} = \mathbb{C}_{3\pi/2}$. Since f and g agree on the region $\mathcal{T} = \{z : \operatorname{Re} z > 0\} \subseteq \mathcal{R} \cap \mathcal{S}$, as you saw in the discussion following Exercise 5.4, we deduce that f and g are direct analytic continuations of each other.

Exercise 5.5

Prove that each of the following pairs of analytic functions f and g are direct analytic continuations of each other.

(a) $f(z) = \displaystyle\sum_{n=1}^{\infty} n z^{n-1} \quad (|z| < 1), \quad g(z) = \frac{1}{(1-z)^2} \quad (z \in \mathbb{C} - \{1\})$

(b) $f(z) = \operatorname{Log}_{2\pi}(z) \quad (z \in \mathbb{C}_{2\pi}), \quad g(z) = \operatorname{Log}_{3\pi/2}(z) \quad (z \in \mathbb{C}_{3\pi/2})$

Often we are given a function f that is analytic on a region $\mathcal{R}$ and we are required to find a direct analytic continuation of f from $\mathcal{R}$ to some overlapping region $\mathcal{S}$. The following exercise gives you an opportunity to try this.

Exercise 5.6

Use Example 5.2(b) to find a direct analytic continuation of the following function f from its domain to another region:

$$f(z) = \sqrt{z} \quad (z \in \mathbb{C}_\pi).$$

(*Hint*: Recall that $\sqrt{z} = \exp\left(\frac{1}{2}\operatorname{Log} z\right)$, for $z \neq 0$.)

5.3 More improper integrals

In this subsection we show how the Residue Theorem can be combined with a simple direct analytic continuation to evaluate improper integrals of the forms

$$\int_0^\infty \frac{p(t)}{q(t)} \log t \, dt \quad \text{and} \quad \int_0^\infty \frac{p(t)}{q(t)} t^a \, dt,$$

where $0 < a < 1$ and p and q are polynomial functions such that the degree of q exceeds that of p by at least two, and any poles of p/q on the *non-negative real axis* (the positive real axis together with the origin) are simple. Integrals of this type may be 'improper at 0' if, for example, p/q has a pole at 0, so we must define what such integrals mean.

Definition

Let f be a function that is continuous on the interval $(0, \infty)$. Then the **improper integral** $\displaystyle\int_0^\infty f(t)\, dt$ is

$$\int_0^\infty f(t)\, dt = \lim_{\varepsilon \to 0} \int_\varepsilon^1 f(t)\, dt + \lim_{r \to \infty} \int_1^r f(t)\, dt,$$

provided that both limits exist.

As usual, the limit $\lim\limits_{\varepsilon \to 0}$ is taken through positive values of ε.

The definition is illustrated in Figure 5.7, where

$$\int_\varepsilon^1 f(t)\, dt \quad \text{and} \quad \int_1^r f(t)\, dt$$

are the areas of the left and right shaded regions, respectively.

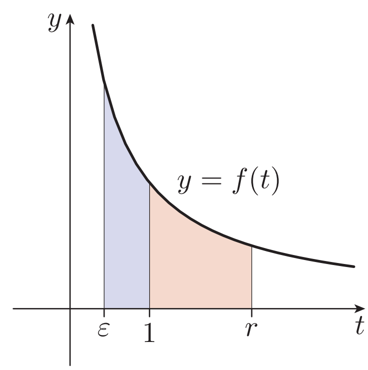


Figure 5.7 Areas under the graph of $y = f(t)$ between ε and 1, and between 1 and r

We remark that the limit of integration 1 is purely a convenient choice; replacing 1 by 2 (or any other positive number) in the definition does not affect the value of $\displaystyle\int_0^\infty f(t)\, dt$.

There are two difficulties in evaluating integrals of one of the two forms listed at the start of the subsection. The first is that an integral of one of these types is taken only along the non-negative real axis (between 0 and ∞), so the standard semicircular contour used in Section 3 is not always appropriate. The other difficulty is the use of the expressions $\log t$ and t^a, which suggest that the principal logarithm function Log will be needed. This function is analytic only on the cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$, which does not include all of our standard semicircular contour. As you will see, we circumvent this difficulty by using various analytic continuations of Log.

Before we describe the procedure for evaluating these integrals, however, you should attempt the following exercise, the results from which will be needed in the examples and exercises that follow.

Exercise 5.7

(a) Show that

$$\int_1^r \frac{1}{t}\,dt \leq \int_1^r \frac{1}{\sqrt{t}}\,dt, \quad \text{for } r > 1,$$

and deduce that

$$\log r \leq 2\sqrt{r}, \quad \text{for } r > 1.$$

(b) Use the second of these two inequalities to prove that

(i) $\dfrac{\log r}{r} \to 0$ as $r \to \infty$

(ii) $\varepsilon \log \varepsilon \to 0$ as $\varepsilon \to 0$ through positive values of ε.

Let us now consider how to use the Residue Theorem to evaluate the improper integral

$$\int_0^\infty \frac{\log t}{t^2+4}\,dt.$$

Observe that $\log t$ is defined only when t is positive (not at $t = 0$), so this integral is 'improper at 0' as well as 'improper at ∞'; it is given by

$$\int_0^\infty \frac{\log t}{t^2+4}\,dt = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{\log t}{t^2+4}\,dt + \lim_{r \to \infty} \int_1^r \frac{\log t}{t^2+4}\,dt.$$

We now carry out the five steps for evaluating improper integrals from Section 3.

1. Find a suitable contour integral

To specify a suitable contour integral, we first need to find a complex function that takes the value $(\log t)/(t^2+4)$, for each positive number t. We might try the complex function with rule $(\operatorname{Log} z)/(z^2+4)$; however, because Log is not analytic at points on the negative real axis, this choice of logarithm function is unsuitable for the contour we wish to use. Instead we define

$$f(z) = \frac{\operatorname{Log}_{3\pi/2}(z)}{z^2+4},$$

which is analytic on the cut plane $\mathbb{C}_{3\pi/2}$, except for a simple pole at $2i$, and which satisfies $f(t) = (\log t)/(t^2+4)$, for $t > 0$. Notice that $-2i$, the other zero of the polynomial z^2+4, does *not* belong to the cut plane $\mathbb{C}_{3\pi/2}$: it lies on the cut, as shown in Figure 5.8.

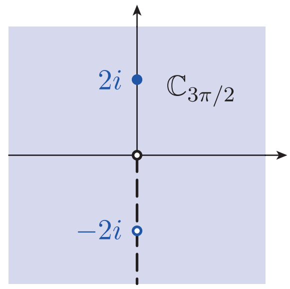


Figure 5.8 The cut plane $\mathbb{C}_{3\pi/2}$

We choose the contour integral

$$I = \int_\Gamma f(z)\,dz,$$

where $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ is the contour shown in Figure 5.9, for positive numbers $\varepsilon < 1$ and $r > 2$, chosen so that Γ encloses the pole of f at $2i$.

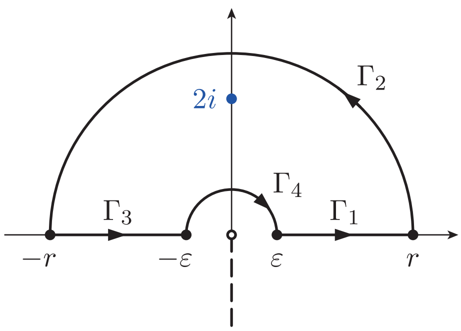


Figure 5.9 The contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$

2. Apply the Residue Theorem

To apply the Residue Theorem on the simply connected region $\mathbb{C}_{3\pi/2}$, we need to find the residue of f at $2i$. Since

$$f(z) = \frac{\operatorname{Log}_{3\pi/2}(z)}{(z-2i)(z+2i)},$$

we can find $\operatorname{Res}(f, 2i)$ using the Cover-up Rule. First observe that

$$\operatorname{Log}_{3\pi/2}(2i) = \log|2i| + i\operatorname{Arg}_{3\pi/2}(2i) = \log 2 + i\pi/2,$$

and hence

$$\operatorname{Res}(f, 2i) = \frac{\operatorname{Log}_{3\pi/2}(2i)}{2i+2i} = \frac{\log 2 + i\pi/2}{4i}.$$

So, by the Residue Theorem,

$$I = 2\pi i \operatorname{Res}(f, 2i) = 2\pi i \times \frac{\log 2 + i\pi/2}{4i} = \frac{\pi}{2}\log 2 + \frac{i\pi^2}{4}.$$

3. Split up the integral

Next we separate the integral into its constituent parts, to give

$$I = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \int_{\Gamma_3} f(z)\,dz + \int_{\Gamma_4} f(z)\,dz. \tag{5.5}$$

Substituting the positive real variable t for z in the Γ_1 integral gives

$$\int_{\Gamma_1} f(z)\,dz = \int_{\varepsilon}^{r} \frac{\log t}{t^2+4}\,dt.$$

Since z also takes real values on Γ_3, we can again substitute t for z in the Γ_3 integral. However, we must be careful here to make sure we use the correct values of the logarithm. If $t < 0$, then $\operatorname{Arg}_{3\pi/2}(t) = \pi$, so

$$\operatorname{Log}_{3\pi/2}(t) = \log|t| + i\pi.$$

Thus, substituting t for z, we see that

$$\int_{\Gamma_3} f(z)\,dz = \int_{-r}^{-\varepsilon} \frac{\log|t| + i\pi}{t^2+4}\,dt.$$

The integrand $(\log|t| + i\pi)/(t^2+4)$ represents an *even* function, which implies that the integral of this function from $-r$ to $-\varepsilon$ is equal to the integral from ε to r. (This can be established by making the substitution $u = -t$, as in the proof of Theorem 3.2(b).) Hence

$$\int_{\Gamma_3} f(z)\,dz = \int_{\varepsilon}^{r} \frac{\log t + i\pi}{t^2+4}\,dt = \int_{\varepsilon}^{r} \frac{\log t}{t^2+4}\,dt + i\pi\int_{\varepsilon}^{r} \frac{1}{t^2+4}\,dt.$$

4. Apply the Estimation Theorem

We must now obtain upper estimates for the moduli of the integrals along the semicircles Γ_2 and Γ_4.

Observe that for $z \in \Gamma_2$, we have $|z| = r$ and $0 \le \operatorname{Arg}_{3\pi/2}(z) \le \pi$, so

$$\begin{aligned} |\operatorname{Log}_{3\pi/2}(z)| &= |\log|z| + i\operatorname{Arg}_{3\pi/2}(z)| \\ &\le |\log r| + |\operatorname{Arg}_{3\pi/2}(z)| \le \log r + \pi, \end{aligned} \tag{5.6}$$

by the Triangle Inequality. Also, for $z \in \Gamma_2$, we have

$$|z^2 + 4| \ge |z^2| - 4 = r^2 - 4, \tag{5.7}$$

by the backwards form of the Triangle Inequality (since $r > 2$). Hence

$$|f(z)| = \left|\frac{\operatorname{Log}_{3\pi/2}(z)}{z^2+4}\right| \le \frac{\log r + \pi}{r^2-4},$$

for $z \in \Gamma_2$. Since f is continuous on Γ_2, and Γ_2 has length πr, we can apply the Estimation Theorem to give

$$\left|\int_{\Gamma_2} f(z)\,dz\right| \le \frac{\log r + \pi}{r^2-4} \times \pi r = \frac{\pi r(\log r + \pi)}{r^2-4},$$

for $r > 2$.

Reasoning in a similar way but with ε in place of r, we can obtain an estimate for the Γ_4 integral. However, there is an important difference this time, namely that $0 < \varepsilon < 1$, which implies that $\log \varepsilon$ is negative (so $|\log \varepsilon| = -\log \varepsilon$). Consequently, for $z \in \Gamma_4$, inequalities (5.6) and (5.7) are replaced with

$$|\mathrm{Log}_{3\pi/2}(z)| \leq -\log \varepsilon + \pi \quad \text{and} \quad |z^2 + 4| \geq 4 - \varepsilon^2.$$

Putting these together, and applying the Estimation Theorem, we see that

$$\left| \int_{\Gamma_4} f(z)\,dz \right| \leq \frac{\pi - \log \varepsilon}{4 - \varepsilon^2} \times \pi\varepsilon = \frac{\pi\varepsilon(\pi - \log \varepsilon)}{4 - \varepsilon^2},$$

for $0 < \varepsilon < 1$.

5. Take limits

Using the value of I found in step 2, and the real integrals discussed in step 3, we can write equation (5.5) as

$$2\int_{\varepsilon}^{r} \frac{\log t}{t^2 + 4}\,dt + i\pi \int_{\varepsilon}^{r} \frac{1}{t^2 + 4}\,dt + \int_{\Gamma_2} f(z)\,dz + \int_{\Gamma_4} f(z)\,dz = \frac{\pi}{2}\log 2 + \frac{i\pi^2}{4}.$$

Let us now take limits as $r \to \infty$ and as $\varepsilon \to 0$ of each expression in this equation. By step 4, the Γ_2 integral tends to 0 as $r \to \infty$ because

$$\left| \int_{\Gamma_2} f(z)\,dz \right| \leq \frac{\pi r(\log r + \pi)}{r^2 - 4} = \pi \frac{(\log r)/r + \pi/r}{1 - 4/r^2},$$

and this final expression tends to 0 as $r \to \infty$ by Exercise 5.7(b)(i) and the Combination Rules for limits of functions (Theorem 3.1). Likewise, by step 4, the Γ_4 integral tends to 0 as $\varepsilon \to 0$ because

$$\left| \int_{\Gamma_4} f(z)\,dz \right| \leq \frac{\pi\varepsilon(\pi - \log \varepsilon)}{4 - \varepsilon^2} = \pi \frac{\pi\varepsilon - \varepsilon \log \varepsilon}{4 - \varepsilon^2},$$

and this final expression tends to 0 as $\varepsilon \to 0$ by Exercise 5.7(b)(ii) and the Combination Rules for limits of functions.

Therefore, on letting $r \to \infty$ and $\varepsilon \to 0$, we obtain

$$2\int_{0}^{\infty} \frac{\log t}{t^2 + 4}\,dt + i\pi \int_{0}^{\infty} \frac{1}{t^2 + 4}\,dt = \frac{\pi}{2}\left(\log 2 + i\frac{\pi}{2}\right).$$

Equating real parts, we see that

$$2\int_{0}^{\infty} \frac{\log t}{t^2 + 4}\,dt = \frac{\pi}{2}\log 2, \quad \text{so} \quad \int_{0}^{\infty} \frac{\log t}{t^2 + 4}\,dt = \frac{\pi}{4}\log 2.$$

Notice that we can equate imaginary parts too, to obtain another integral, for free:

$$\pi \int_{0}^{\infty} \frac{1}{t^2 + 4}\,dt = \frac{\pi^2}{4}, \quad \text{so} \quad \int_{0}^{\infty} \frac{1}{t^2 + 4}\,dt = \frac{\pi}{4}.$$

However, this integral could be obtained by more elementary means (the integrand has a primitive $t \longmapsto \frac{1}{2}\tan^{-1}(t/2)$).

We can use a similar strategy to evaluate the improper integral

$$\int_0^\infty \frac{t^{-1/2}}{t^2+1}\,dt.$$

This is another improper integral that is 'improper at 0' as well as 'improper at ∞' because the term $t^{-1/2} = 1/\sqrt{t}$ grows arbitrarily large as t approaches 0.

We need to find a complex function f such that $f(t) = t^{-1/2}/(t^2+1)$, for $t > 0$. It is tempting to use the function $z \longmapsto z^{-1/2}/(z^2+1)$, but this is unsuitable for the semicircular contour we wish to use because $z^{-1/2}$ is not continuous on the negative real axis. Instead we recall that

$$t^{-1/2} = \exp\left(-\tfrac{1}{2}\log t\right), \quad \text{for } t > 0,$$

and, motivated by the earlier example, we define

$$f(z) = \frac{\exp\left(-\frac{1}{2}\operatorname{Log}_{3\pi/2}(z)\right)}{z^2+1},$$

which does indeed satisfy

$$f(t) = \frac{t^{-1/2}}{t^2+1}, \quad \text{for } t > 0.$$

We can now proceed in much the same way as the earlier example; the details are in the solution to the following exercise.

Exercise 5.8

Evaluate the improper integral

$$\int_0^\infty \frac{t^{-1/2}}{t^2+1}\,dt.$$

We have now evaluated

$$\int_0^\infty \frac{\log t}{t^2+4}\,dt \quad \text{and} \quad \int_0^\infty \frac{t^{-1/2}}{t^2+1}\,dt,$$

by using a semicircular contour, indented at the origin. In order for this method to work, it was essential that the rational functions in these integrands were even functions. When faced with an integral such as

$$\int_0^\infty \frac{t^{1/4}}{t^2+t}\,dt,$$

in which the rational function in the integrand is *not* an even function, some other method must be found.

One possible approach is to make the rational function in the integrand even by using the preliminary substitution $u = \sqrt{t}$, so $t = u^2$, $dt = 2u\,du$.

In the example above, we obtain

$$\begin{aligned}\int_0^\infty \frac{t^{1/4}}{t^2+t}\,dt &= \int_0^\infty \frac{u^{1/2}}{u^4+u^2}2u\,du\\ &= 2\int_0^\infty \frac{u^{-1/2}}{u^2+1}\,du\\ &= 2\left(\frac{\pi}{\sqrt{2}}\right) = \sqrt{2}\pi,\end{aligned}$$

by Exercise 5.8. However, this strategy will not work for all rational functions, so instead we introduce an alternative method based on the contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ shown in Figure 5.10, where $0 < \varepsilon < 1 < r$. Here Γ_1 and Γ_3 are the same interval $[\varepsilon, r]$ of the positive real axis, traversed in opposite directions, and Γ_2 and Γ_4 are circles with centre 0. Thus the contour Γ is closed, but not simple-closed.

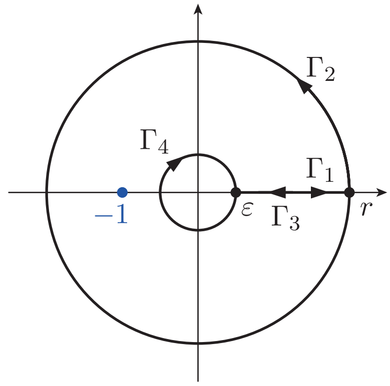


Figure 5.10 The contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$

We then introduce the function

$$f(z) = \frac{\exp\left(\frac{1}{4}\operatorname{Log}_{2\pi}(z)\right)}{z^2+z} \qquad (z \in \mathbb{C}_{2\pi} - \{-1\}), \tag{5.8}$$

which is analytic on the region $\mathbb{C}_{2\pi} - \{-1\}$, and extend the definition of f to Γ_1 and Γ_3 as follows:

$$f(z) = \begin{cases} \dfrac{\exp\left(\frac{1}{4}\operatorname{Log} z\right)}{z^2+z} = \dfrac{z^{1/4}}{z^2+z}, & z \in \Gamma_1,\\[2ex] \dfrac{\exp\left(\frac{1}{4}\operatorname{Log}_{2\pi}(z)\right)}{z^2+z} = \dfrac{z^{1/4}e^{(1/4)2\pi i}}{z^2+z}, & z \in \Gamma_3. \end{cases} \tag{5.9}$$

For the second of these formulas, notice that $\operatorname{Log}_{2\pi}$ has domain $\mathbb{C} - \{0\}$ so it *is* defined on Γ_3, even though it is not analytic (or even continuous) on $\mathbb{C} - \{0\}$.

Strictly speaking, this extension of f to Γ_1 and Γ_3 is ambiguous because, as sets, Γ_1 and Γ_3 are equal, so the extended f is not a function. However, if you follow z round the contour Γ, using the values of f in equation (5.9) on Γ_1 and Γ_3, then $f(z)$ varies continuously on Γ and behaves like the 'boundary values' of the function f defined by equation (5.8).

Assuming that the conclusion of the Residue Theorem holds in this situation, we deduce that

$$\begin{aligned}\int_\Gamma f(z)\,dz &= 2\pi i \operatorname{Res}(f,-1)\\ &= \left(2\pi i \exp\left(\tfrac{1}{4}\operatorname{Log}_{2\pi}(-1)\right)\right)/(-1)\\ &= -2\pi i \exp\left(\tfrac{1}{4}(\log|-1| + i\operatorname{Arg}_{2\pi}(-1))\right)\\ &= -2\pi i e^{\pi i/4},\end{aligned} \tag{5.10}$$

where we have used the Cover-up Rule to find the residue of f at -1, the only pole of f 'inside' Γ. (Because Γ is not simple-closed, it does not really have an 'inside'. However, in the present situation we regard the 'inside' as the subset of $\mathbb{C}$ that lies on your left as you traverse Γ, as shown in Figure 5.11.)

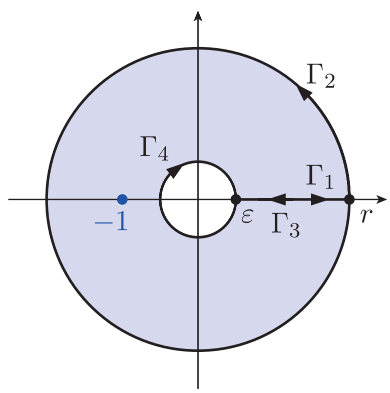


Figure 5.11 The 'inside' of the contour Γ

As in Exercise 5.8, we can show that

$$\int_{\Gamma_2} f(z)\,dz \to 0 \text{ as } r \to \infty \quad \text{and} \quad \int_{\Gamma_4} f(z)\,dz \to 0 \text{ as } \varepsilon \to 0. \tag{5.11}$$

Also, equation (5.9) implies that

$$\int_{\Gamma_1} f(z)\,dz = \int_{\varepsilon}^{r} \frac{t^{1/4}}{t^2+t}\,dt \tag{5.12}$$

and

$$\int_{\Gamma_3} f(z)\,dz = e^{\pi i/2}\int_{r}^{\varepsilon} \frac{t^{1/4}}{t^2+t}\,dt = -e^{\pi i/2}\int_{\varepsilon}^{r} \frac{t^{1/4}}{t^2+t}\,dt. \tag{5.13}$$

Letting $r \to \infty$ and $\varepsilon \to 0$, we deduce from equations (5.10)–(5.13) that

$$\left(1 - e^{\pi i/2}\right)\int_0^\infty \frac{t^{1/4}}{t^2+t}\,dt = -2\pi i e^{\pi i/4},$$

so

$$\begin{aligned}\int_0^\infty \frac{t^{1/4}}{t^2+t}\,dt &= \frac{-2\pi i e^{\pi i/4}}{1 - e^{\pi i/2}}\\ &= \frac{2\pi i}{e^{\pi i/4} - e^{-\pi i/4}}\\ &= \frac{\pi}{\sin \pi/4} = \sqrt{2}\pi,\end{aligned}$$

as obtained above.

Rather than discuss the justification of this method of evaluation in detail, we simply state (and do not prove) a general result that can be obtained by this method. As with the corresponding results in Section 3, it is possible to allow simple poles on the positive real axis (by using the Round-the-Pole Lemma).

Theorem 5.3

Let p and q be polynomial functions such that

- the degree of q exceeds the degree of p by at least two
- any poles of p/q on the non-negative real axis are simple.

Then, for $0 < a < 1$,

$$\int_0^\infty \frac{p(t)}{q(t)} t^a\,dt = -\left(\pi e^{-\pi a i} \operatorname{cosec} \pi a\right)S - (\pi \cot \pi a)T,$$

where S is the sum of the residues of the function

$$f_1(z) = \frac{p(z)}{q(z)} \exp(a \operatorname{Log}_{2\pi}(z))$$

in $\mathbb{C}_{2\pi}$, and T is the sum of the residues of the function

$$f_2(z) = \frac{p(z)}{q(z)} \exp(a \operatorname{Log} z)$$

on the positive real axis.

Note that the non-negative real axis includes 0 but the positive real axis does not.

Exercise 5.9

Use Theorem 5.3 to show that

$$\int_0^\infty \frac{t^a}{t^2 - t}\,dt = -\pi \cot \pi a, \quad \text{for } 0 < a < 1.$$

Remark

As well as being used in the evaluation of integrals of the form

$$\int_0^\infty \frac{p(t)}{q(t)} t^a\,dt,$$

the contour in Figure 5.10 can also be used to evaluate a wide variety of other integrals, including those of the form

$$\int_0^\infty \frac{p(t)}{q(t)} \log t\,dt.$$

5.4 Indirect analytic continuation

Having obtained a direct analytic continuation of a given analytic function f with domain $\mathcal{R}$ to an analytic function g with domain $\mathcal{S}$, it is natural to attempt to carry this process further by finding an analytic function h with domain $\mathcal{T}$ that is a direct analytic continuation of g but is not a direct analytic continuation of f (see Figure 5.12).

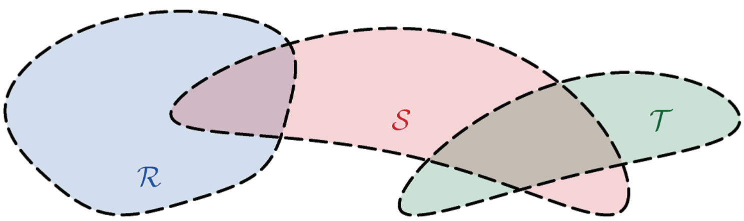


Figure 5.12 Intersecting regions $\mathcal{R}$ and $\mathcal{S}$, and $\mathcal{S}$ and $\mathcal{T}$

For example, if

$$\begin{aligned} f_1(z) &= \mathrm{Log}_\pi(z) \quad (z \in \mathbb{C}_\pi), \\ f_2(z) &= \mathrm{Log}_{2\pi}(z) \quad (z \in \mathbb{C}_{2\pi}), \\ f_3(z) &= \mathrm{Log}_{3\pi}(z) \quad (z \in \mathbb{C}_{3\pi}), \end{aligned}$$

then

$$\begin{aligned} f_2(z) &= f_1(z), \quad \text{for } \operatorname{Im} z > 0, \\ f_3(z) &= f_2(z), \quad \text{for } \operatorname{Im} z < 0, \end{aligned}$$

(see Figure 5.13).

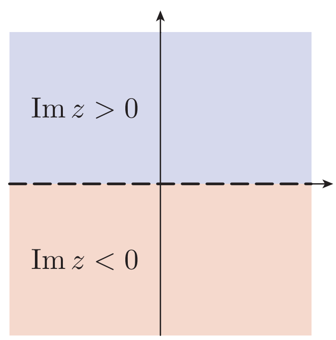


Figure 5.13 The upper and lower half-planes

Since the region $\{z : \operatorname{Im} z > 0\}$ is a subset of $\mathbb{C}_\pi \cap \mathbb{C}_{2\pi}$, we see that f_2 is a direct analytic continuation of f_1 from $\mathbb{C}_\pi$ to $\mathbb{C}_{2\pi}$. Similarly, f_3 is a direct analytic continuation of f_2 from $\mathbb{C}_{2\pi}$ to $\mathbb{C}_{3\pi}$. But

$$f_3(z) = f_1(z) + 2\pi i, \quad \text{for } z \in \mathbb{C}_\pi = \mathbb{C}_{3\pi},$$

so f_3 is *not* a direct analytic continuation of f_1. However, there is clearly a sense in which f_3 is some kind of an 'analytic continuation' of f_1, so we extend our definition accordingly. To emphasise the importance of domains in this definition, we will use the notation $(f, \mathcal{R})$ to denote an analytic function f whose domain is the region $\mathcal{R}$. (In some texts, the pair $(f, \mathcal{R})$ is called a *function element*.)

Definitions

The finite sequence of functions

$$(f_1, \mathcal{R}_1),\ (f_2, \mathcal{R}_2),\ \ldots,\ (f_n, \mathcal{R}_n)$$

is called a **chain of functions** if $(f_{k+1}, \mathcal{R}_{k+1})$ is a direct analytic continuation of $(f_k, \mathcal{R}_k)$, for $k = 1, 2, \ldots, n-1$.

Any two functions of a chain of functions are said to be **analytic continuations** of each other. If the two functions are not direct analytic continuations of each other, then they are said to be **indirect analytic continuations** of each other.

A chain of functions is **closed** if $\mathcal{R}_1 = \mathcal{R}_n$.

Note that for a closed chain of functions, the function f_1 may or may not be equal to f_n.

Two chains of functions are illustrated in Figures 5.14 and 5.15.

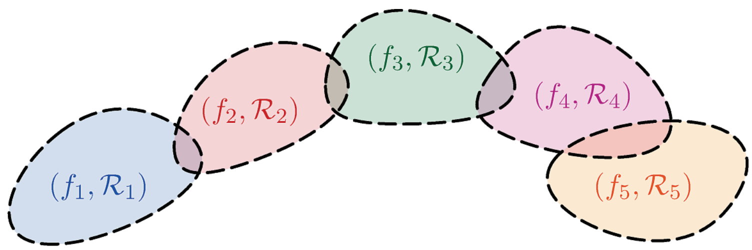


Figure 5.14 A chain of functions

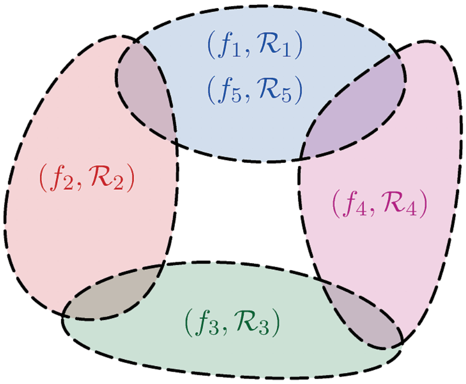


Figure 5.15 A closed chain of functions, with $\mathcal{R}_1 = \mathcal{R}_5$

For an example of indirect analytic continuation, consider the functions

$$f_1(z) = \operatorname{Log}_\pi(z) \quad (z \in \mathbb{C}_\pi) \quad \text{and} \quad f_3(z) = \operatorname{Log}_{3\pi}(z) \quad (z \in \mathbb{C}_{3\pi}),$$

discussed earlier. The sequence $(f_1, \mathbb{C}_\pi), (f_2, \mathbb{C}_{2\pi}), (f_3, \mathbb{C}_{3\pi})$, where

$$f_k(z) = \operatorname{Log}_{k\pi}(z), \quad \text{for } k = 1, 2, 3,$$

is a closed chain of functions because $\mathbb{C}_{3\pi} = \mathbb{C}_\pi$, even though $f_1 \neq f_3$ on this set.

Exercise 5.10

Consider the functions

$$f_k(z) = \exp\left(\tfrac{1}{2} \operatorname{Log}_{k\pi}(z)\right) \quad (z \in \mathbb{C}_{k\pi}),$$

for $k \in \mathbb{Z}$. Notice that $f_1(z) = \sqrt{z}$, for $z \in \mathbb{C}_\pi$.

(a) Show that $(f_1, \mathbb{C}_\pi), (f_2, \mathbb{C}_{2\pi}), (f_3, \mathbb{C}_{3\pi})$ is a closed chain of functions, but $f_1 \neq f_3$.

(b) Show that $(f_1, \mathbb{C}_\pi), (f_2, \mathbb{C}_{2\pi}), (f_3, \mathbb{C}_{3\pi}), (f_4, \mathbb{C}_{4\pi}), (f_5, \mathbb{C}_{5\pi})$ is a closed chain of functions, and $f_1 = f_5$.

Exercise 5.11

Let $D_1 = \{z : |z| < 1\}$ and $D_2 = \{z : |z - 2| < 1\}$. Show that the functions

$$f(z) = \sum_{n=0}^{\infty} z^n \quad (z \in D_1) \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} (-1)^{n+1} (z-2)^n \quad (z \in D_2)$$

are indirect analytic continuations of each other.

(*Hint*: Use the result of Example 5.1.)

Further exercises

Exercise 5.12

Show that the analytic function

$$g(z) = -\frac{1}{z} - 1 \quad (z \in \mathbb{C} - \{0\})$$

is a direct analytic continuation of the analytic function

$$f(z) = (z+1) + (z+1)^2 + (z+1)^3 + \cdots \quad (|z+1| < 1).$$

Exercise 5.13

Let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ $(|z| < 1)$.

(a) Prove that

$$f'(z) = \frac{-\operatorname{Log}(1-z)}{z}, \quad \text{for } 0 < |z| < 1.$$

(b) Deduce that the function

$$g(z) = \begin{cases} \dfrac{-\operatorname{Log}(1-z)}{z}, & z \in \mathbb{C} - \{x \in \mathbb{R} : x = 0 \text{ or } x \geq 1\}, \\ 1, & z = 0, \end{cases}$$

is a direct analytic continuation of f' from $\{z : |z| < 1\}$ to $\mathbb{C} - \{x \in \mathbb{R} : x \geq 1\}$.

(c) Use the Primitive Theorem (Theorem 5.3 of Unit B2) to show that there is a direct analytic continuation of f from $\{z : |z| < 1\}$ to a larger region.

Exercise 5.14

Use Theorem 5.3 to evaluate the following improper integral:

$$\int_0^{\infty} \frac{t^{3/2}}{(t^2+1)(t-1)}\, dt.$$

Exercise 5.15

Show that the functions

$$f(z) = \sum_{n=0}^{\infty} z^n \quad (|z| < 1) \quad \text{and} \quad g(z) = -\sum_{n=1}^{\infty} z^{-n} \quad (|z| > 1)$$

are indirect analytic continuations of each other.

Complete analytic functions and Riemann surfaces

Given an analytic function $(f, \mathcal{R})$, the collection of all possible analytic continuations of $(f, \mathcal{R})$ is called the **complete analytic function** of $(f, \mathcal{R})$. This name is rather misleading because, as defined, a complete analytic function is not actually a function, but a *set* of functions related to each other by analytic continuation. However, in his doctoral dissertation of 1851, the distinguished German mathematician Bernhard Riemann (1826–1866) described a remarkable geometric method for interpreting complete analytic functions in which they do become genuine functions.

Bernhard Riemann

To demonstrate Riemann's idea, let us outline the procedure for obtaining the complete analytic function of the principal logarithm function $(\mathrm{Log}, \mathbb{C}_\pi)$. We define

$$f_k(z) = \mathrm{Log}_{k\pi}(z), \quad \text{for } k \in \mathbb{Z},$$

so $f_1(z) = \mathrm{Log}_\pi z = \mathrm{Log}\, z$. Then the sequence $(f_k, \mathbb{C}_{k\pi})$, $k \in \mathbb{Z}$, is an *infinite* chain of functions, because f_k and f_{k+1} are

- equal on the upper half-plane $\{z : \mathrm{Im}\, z > 0\}$ if k is odd
- equal on the lower half-plane $\{z : \mathrm{Im}\, z < 0\}$ if k is even

(as we observed earlier for $k = 1$ and $k = 2$).

Think of the cut planes $\ldots, \mathbb{C}_{-\pi}, \mathbb{C}_0, \mathbb{C}_\pi, \mathbb{C}_{2\pi}, \ldots$ as separate planes stacked one above the other, with

- a slit along the negative real axis if k is odd
- a slit along the positive real axis if k is even,

as illustrated in Figure 5.16(a). Next, following Riemann's imaginative reasoning we 'paste' the upper half of $\mathbb{C}_{k\pi}$ to the upper half of $\mathbb{C}_{(k+1)\pi}$ when k is odd, since f_k and f_{k+1} are equal on the upper half-plane when k is odd. Similarly, when k is even, we 'paste' the lower half of $\mathbb{C}_{k\pi}$ to the lower half of $\mathbb{C}_{(k+1)\pi}$, as illustrated in Figure 5.16(b).

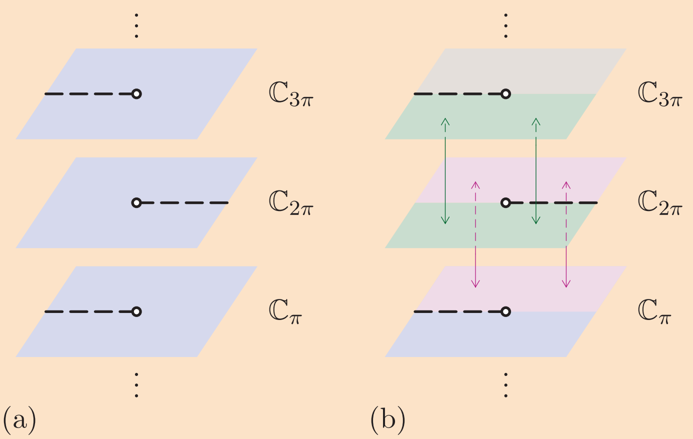


Figure 5.16 (a) The cut planes $\mathbb{C}_\pi, \mathbb{C}_{2\pi}, \mathbb{C}_{3\pi}$ (b) Pasting $\mathbb{C}_\pi$ to $\mathbb{C}_{2\pi}$ and $\mathbb{C}_{2\pi}$ to $\mathbb{C}_{3\pi}$

We now pull the glued planes in a direction perpendicular to the planes, to form a concertina-like surface.

To be more precise, we lift vertically each ray of points with argument θ to a height θ. Part of the resulting surface S is shown in Figure 5.17; it has equation $s = \theta$, where θ is any argument of $z = x + iy$. In fact, we have encountered this surface before, in Figure 2.2 of Unit A2.

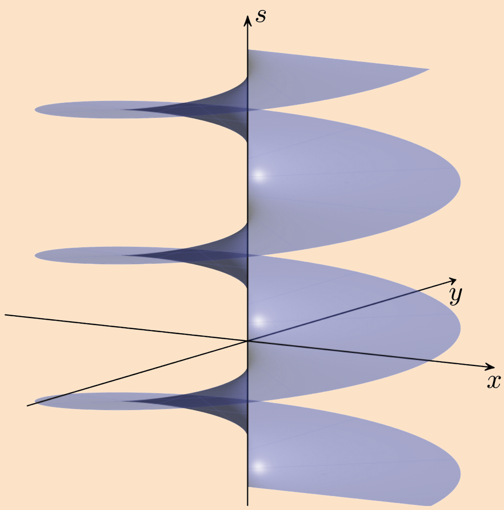


Figure 5.17 The Riemann surface S for the complete analytic function of Log

Let us define a function $f\colon S \longrightarrow \mathbb{C}$ by the rule

$$f(x, y, \theta) = \log|z| + i\theta,$$

where θ is any argument of $z = x + iy$.

As we know, the expression $\log|z| + i\theta$ is *not* the rule of a function on $\mathbb{C} - \{0\}$, because each complex number has infinitely many arguments, but f *is* a function on S, because each argument θ of z corresponds to a different point on the surface S. What is more, if $z \in \mathbb{C}_{k\pi}$, then $\theta = \operatorname{Arg}_{k\pi}(z)$ satisfies $(k-2)\pi < \theta < k\pi$, so

$$f(x, y, \theta) = \log|z| + i\theta = \operatorname{Log}_{k\pi}(z) = f_k(z).$$

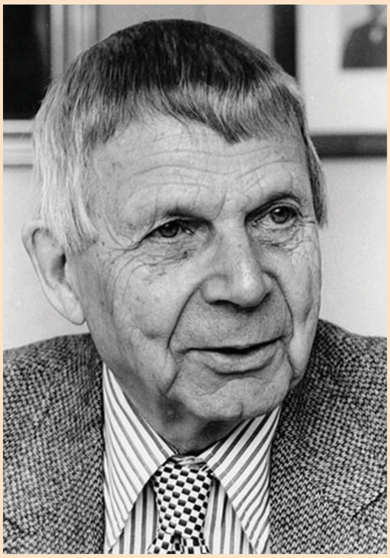
To summarise, we have constructed a surface S by pasting together the sets $\mathbb{C}_{k\pi}$, and we have defined a function $f\colon S \longrightarrow \mathbb{C}$ that coincides with f_k on each layer $\mathbb{C}_{k\pi}$ of the surface. It can be shown that any analytic continuation of $(\operatorname{Log}, \mathbb{C}_\pi)$ can be thought of as a restriction of the function (f, S), and consequently (f, S) can be considered to be the complete analytic function of $(\operatorname{Log}, \mathbb{C}_\pi)$.

The surface S is called the **Riemann surface** for this complete analytic function.

The theory of Riemann surfaces can be taken much further and placed on a sound theoretical basis. It is possible to transfer many of the ideas of complex analysis from the complex plane to a general Riemann surface, and thereby gain insight into functions such as $z \longmapsto \operatorname{Log} z$ and $z \longmapsto \sqrt{z}$, which have analytic continuations that cannot be represented by ordinary analytic functions with domains in the complex plane.

Over the past century, the subject of Riemann surfaces has flourished, and deep connections have been uncovered with other mathematical disciplines. One of the pioneers in this area of study was the Finnish complex analyst Lars Valerian Ahlfors (1907–1996), whose profound geometric insight led to advances in classifying and understanding Riemann surfaces. In 1936 he was awarded the Fields Medal for his work in complex analysis and Riemann surfaces. (The Fields medal, awarded every four years to two to four mathematicians under the age of forty, is often thought of as the 'Nobel Prize for mathematics'.)

More recently, the Iranian mathematician Maryam Mirzakhani (1977–2017) also received a Fields Medal for her work on Riemann surfaces, in 2014. She was the first woman, and the first Iranian, to receive this illustrious award.



Lars Valerian Ahlfors

Maryam Mirzakhani

Solutions to exercises

Solution to Exercise 1.1

(a) The Laurent series about 0 for f is

$$\frac{1}{z^2} - 3 = \cdots + \frac{0}{z^3} + \frac{1}{z^2} + \frac{0}{z} - 3 + 0z + \cdots,$$

so $\operatorname{Res}(f,0) = 0$.

(b) Since the function $f(z) = 1/(z-1)$ is analytic at 0, the Laurent series about 0 for f is a Taylor series. Hence $\operatorname{Res}(f,0) = 0$.

(c) The Laurent series about 0 for f is

$$\begin{aligned}\frac{\cos z}{z^3} &= \frac{1}{z^3}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\\ &= \frac{1}{z^3} - \frac{1}{2!\,z} + \frac{z}{4!} - \frac{z^3}{6!} + \cdots,\end{aligned}$$

so $\operatorname{Res}(f,0) = -1/2$.

(d) The Laurent series about 0 for the function $f(z) = z^2 \sin(1/z)$ is

$$\begin{aligned}&z^2\left((1/z) - \frac{(1/z)^3}{3!} + \frac{(1/z)^5}{5!} - \cdots\right)\\ &= z - \frac{1}{3!\,z} + \frac{1}{5!\,z^3} - \cdots,\end{aligned}$$

so $\operatorname{Res}(f,0) = -1/6$.

(Note that f has an essential singularity at 0, not a pole.)

Solution to Exercise 1.2

(a) Observe that

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}.$$

We let $w = z - i$, so $z = i + w$. When $z \neq i, -i$, we have

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{(z-i)(z+i)}\\ &= \frac{1}{w(2i+w)}\\ &= \frac{1}{2iw}\left(1 + \frac{w}{2i}\right)^{-1}.\end{aligned}$$

If $w \neq 0$ and $|w/(2i)| < 1$ (that is, if $0 < |w| < 2$), then we obtain

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{2iw}\left(1 - \left(\frac{w}{2i}\right) + \left(\frac{w}{2i}\right)^2 - \cdots\right)\\ &= \frac{1}{2iw} + \frac{1}{4} - \frac{w}{8i} - \cdots.\end{aligned}$$

The coefficient of w^{-1} in this expression is $1/(2i)$, so

$$\operatorname{Res}(f,i) = \frac{1}{2i} = -\frac{i}{2}.$$

(b) Let $w = z - \pi$, so $z = \pi + w$. We obtain, for $z \neq \pi$,

$$\begin{aligned}\frac{ze^{iz}}{(z-\pi)^2} &= \frac{(\pi+w)e^{i(\pi+w)}}{w^2}\\ &= \frac{e^{i\pi}(\pi+w)}{w^2}\,e^{iw}\\ &= -\left(\frac{\pi+w}{w^2}\right)\left(1 + iw + \frac{(iw)^2}{2!} + \cdots\right)\\ &= -\frac{\pi}{w^2}\left(1 + iw - \frac{w^2}{2} + \cdots\right)\\ &\quad - \frac{1}{w}\left(1 + iw - \frac{w^2}{2} + \cdots\right).\end{aligned}$$

The coefficient of w^{-1} in this expression is $-\pi i - 1$, so $\operatorname{Res}(f,\pi) = -\pi i - 1$.

Solution to Exercise 1.3

(a) Since

$$\lim_{z\to 2i}\left((z-2i) \times \frac{1}{z^2+4}\right) = \lim_{z\to 2i} \frac{1}{z+2i} = \frac{1}{4i},$$

we deduce, by Theorem 1.2, that

$$\operatorname{Res}(f,2i) = \frac{1}{4i} = -\frac{i}{4}.$$

(b) Since

$$\begin{aligned}&\lim_{z\to 1/3}\left(\left(z - \tfrac{1}{3}\right) \times \frac{1}{z^2(1-z)(1-2z)(1-3z)}\right)\\ &= \lim_{z\to 1/3} \frac{-\frac{1}{3}}{z^2(1-z)(1-2z)}\\ &= \frac{-\frac{1}{3}}{\frac{1}{9} \times \frac{2}{3} \times \frac{1}{3}} = -\frac{27}{2},\end{aligned}$$

we deduce, by Theorem 1.2, that

$$\operatorname{Res}\left(f, \tfrac{1}{3}\right) = -27/2.$$

Solution to Exercise 1.4

(a) Let $g(z) = 1$ and $h(z) = 2z^2 + 5iz - 2$. Then g and h are analytic at $-\frac{1}{2}i$. Also,

$$\begin{aligned} h\left(-\tfrac{1}{2}i\right) &= 2\left(-\tfrac{1}{2}i\right)^2 + 5i\left(-\tfrac{1}{2}i\right) - 2 \\ &= -\tfrac{1}{2} + \tfrac{5}{2} - 2 = 0, \end{aligned}$$

and

$$h'\left(-\tfrac{1}{2}i\right) = 4\left(-\tfrac{1}{2}i\right) + 5i = 3i,$$

which is non-zero. Thus the g/h Rule applies, and we have

$$\begin{aligned} \operatorname{Res}\left(f, -\tfrac{1}{2}i\right) &= g\left(-\tfrac{1}{2}i\right)/h'\left(-\tfrac{1}{2}i\right) \\ &= 1/(3i) = -i/3. \end{aligned}$$

(b) Let $g(z) = z + 9$ and $h(z) = (z^2 + 1)(z^2 + 9)$. Then g and h are analytic at $3i$. Also,

$$h(3i) = \left((3i)^2 + 1\right)\left((3i)^2 + 9\right) = -8 \times 0 = 0,$$

and

$$h'(z) = 2z(z^2 + 9) + 2z(z^2 + 1) = 4z(z^2 + 5),$$

so

$$h'(3i) = 4 \times 3i \times \left((3i)^2 + 5\right) = -48i,$$

which is non-zero.

Thus the g/h Rule applies, and we have

$$\begin{aligned} \operatorname{Res}(f, 3i) &= g(3i)/h'(3i) \\ &= (3i + 9)/(-48i) \\ &= -\tfrac{1}{16} + \tfrac{3}{16}i. \end{aligned}$$

Alternatively, we can simplify the calculation of $h(3i)$ and $h'(3i)$ by choosing

$$g(z) = \frac{z+9}{z^2+1} \quad \text{and} \quad h(z) = z^2 + 9.$$

Then g and h are analytic at $3i$. Also,

$$h(3i) = (3i)^2 + 9 = 0,$$

and

$$h'(3i) = 2 \times 3i = 6i,$$

which is non-zero.

Thus the g/h Rule applies, and we have

$$\begin{aligned} \operatorname{Res}(f, 3i) &= g(3i)/h'(3i) \\ &= \left(\frac{3i + 9}{(3i)^2 + 1}\right)\Big/ 6i \\ &= (3i + 9)/(-48i) \\ &= -\tfrac{1}{16} + \tfrac{3}{16}i, \end{aligned}$$

as before.

(c) Let $g(z) = z^3$ and $h(z) = z^4 + 1$. Then g and h are analytic at each given value of α. Also, in each case,

$$h(\alpha) = \alpha^4 + 1 = (-1) + 1 = 0,$$

and $h'(\alpha) = 4\alpha^3$, which is non-zero.

Thus the g/h Rule applies, and we have

$$\begin{aligned} \operatorname{Res}(f, \alpha) &= g(\alpha)/h'(\alpha) \\ &= \alpha^3/(4\alpha^3) \\ &= \tfrac{1}{4}, \end{aligned}$$

for each value of α.

(Note that, in this example, each singularity has the same residue.)

Solution to Exercise 1.5

(a) Let $g(z) = \pi \operatorname{cosec} \pi z$ and $h(z) = 4z^2 - 1$. Then g and h are analytic at $\frac{1}{2}$ and $-\frac{1}{2}$. Also,

$$h\left(\tfrac{1}{2}\right) = 4\left(\tfrac{1}{2}\right)^2 - 1 = 0,$$

and

$$h'\left(\tfrac{1}{2}\right) = 8 \times \tfrac{1}{2} = 4,$$

which is non-zero. Thus the g/h Rule applies, and we have

$$\operatorname{Res}\left(f, \tfrac{1}{2}\right) = (\pi \operatorname{cosec} \pi/2)/4 = \pi/4.$$

Since f is an odd function, we see from Theorem 1.1(a) that

$$\operatorname{Res}\left(f, -\tfrac{1}{2}\right) = \operatorname{Res}\left(f, \tfrac{1}{2}\right) = \pi/4.$$

(b) Let $g(z) = \pi \cot \pi z$ and $h(z) = 4z^2 - 1$. Then g and h are analytic at $\frac{1}{2}$ and $-\frac{1}{2}$, and the calculations of $h\left(\frac{1}{2}\right)$ and $h'\left(\frac{1}{2}\right)$ are as in part (a). Thus the g/h Rule applies, and we have

$$\operatorname{Res}\left(f, \tfrac{1}{2}\right) = (\pi \cot \pi/2)/4 = 0.$$

Since f is an odd function, we see from Theorem 1.1(a) that

$$\operatorname{Res}\left(f, -\tfrac{1}{2}\right) = \operatorname{Res}\left(f, \tfrac{1}{2}\right) = 0.$$

(c) Let $g(z) = \pi \operatorname{cosec} \pi z$ and $h(z) = 4z^2 + 1$. Then g and h are analytic at $\frac{1}{2}i$ and $-\frac{1}{2}i$. Also,

$$h\left(\tfrac{1}{2}i\right) = 4\left(\tfrac{1}{2}i\right)^2 + 1 = 0,$$

and

$$h'\left(\tfrac{1}{2}i\right) = 8 \times \tfrac{1}{2}i = 4i,$$

which is non-zero. Thus the g/h Rule applies, and we have

$$\begin{aligned}\operatorname{Res}\left(f, \tfrac{1}{2}i\right) &= (\pi \operatorname{cosec} \pi i/2)/4i \\ &= \left(\frac{\pi}{i \sinh \pi/2}\right) \Big/ 4i \\ &= -\frac{\pi}{4 \sinh \pi/2}.\end{aligned}$$

Since f is an odd function, we see from Theorem 1.1(a) that

$$\operatorname{Res}\left(f, -\tfrac{1}{2}i\right) = \operatorname{Res}\left(f, \tfrac{1}{2}i\right) = -\frac{\pi}{4 \sinh \pi/2}.$$

(d) Let $g(z) = \pi \cot \pi z$ and $h(z) = 4z^2 + 1$. Then g and h are analytic at $\frac{1}{2}i$ and $-\frac{1}{2}i$, and the calculations of $h\left(\frac{1}{2}i\right)$ and $h'\left(\frac{1}{2}i\right)$ are as in part (c). Thus the g/h Rule applies, and we have

$$\begin{aligned}\operatorname{Res}\left(f, \tfrac{1}{2}i\right) &= (\pi \cot \pi i/2)/4i \\ &= \left(\frac{\pi \cosh \pi/2}{i \sinh \pi/2}\right) \Big/ 4i \\ &= -\frac{\pi \cosh \pi/2}{4 \sinh \pi/2} \\ &= -\frac{\pi}{4} \coth \pi/2.\end{aligned}$$

Since f is an odd function, we see from Theorem 1.1(a) that

$$\operatorname{Res}\left(f, -\tfrac{1}{2}i\right) = \operatorname{Res}\left(f, \tfrac{1}{2}i\right) = -\frac{\pi}{4} \coth \pi/2.$$

Solution to Exercise 1.6

(a) The function

$$f(z) = \frac{z+2}{z^3(z+4)}$$

has a simple pole at the point -4. If we cover up the factor $z + 4$ and evaluate what remains at -4, then we obtain

$$\operatorname{Res}(f, -4) = \frac{-4+2}{(-4)^3 \, (z+4)} = \frac{1}{32},$$

by the Cover-up rule.

(b) We have $f(z) = g(z)/z$, where $g(z) = (\cos z)/e^z$. The function g is analytic at the point 0. Applying the Cover-up Rule, we obtain

$$\operatorname{Res}(f, 0) = g(0) = \frac{\cos 0}{e^0} = 1.$$

(c) First we write

$$f(z) = \frac{-1}{3z^2(1-z)(1-2z)\left(z - \frac{1}{3}\right)}.$$

The function f has a simple pole at the point $\frac{1}{3}$. Applying the Cover-up Rule, we obtain

$$\operatorname{Res}\left(f, \tfrac{1}{3}\right) = \frac{-1}{3 \times \frac{1}{9} \times \frac{2}{3} \times \frac{1}{3}} = -\frac{27}{2}.$$

This agrees with the answer to Exercise 1.3(b).

(d) We have $f(z) = g(z)/z$, where $g(z) = (\sin z)/e^z$. The function g is analytic at the point 0. Applying the Cover-up Rule, we obtain

$$\operatorname{Res}(f, 0) = g(0) = \frac{\sin 0}{e^0} = 0$$

(so g has a removable singularity at 0).

Solution to Exercise 1.7

(a) We have

$$\begin{aligned}f(z) &= \frac{z+2}{z^3(z+4)} \\ &= \frac{1}{4z^3}(z+2)\left(1 + \frac{z}{4}\right)^{-1} \\ &= \frac{1}{4z^3}(z+2)\left(1 - \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 - \left(\frac{z}{4}\right)^3 + \cdots\right),\end{aligned}$$

for $0 < |z| < 4$. Hence

$$\begin{aligned}f(z) = \frac{1}{4z^3}\Bigg(&\left(z - \frac{z^2}{4} + \frac{z^3}{16} - \frac{z^4}{64} + \cdots\right) \\ &+ \left(2 - \frac{z}{2} + \frac{z^2}{8} - \frac{z^3}{32} + \cdots\right)\Bigg),\end{aligned}$$

so

$$\operatorname{Res}(f, 0) = \frac{1}{4}\left(-\frac{1}{4} + \frac{1}{8}\right) = -\frac{1}{32}.$$

(b) First we express $f(z)$ in terms of $z-1$ to give

$$f(z) = \frac{1+e^{2z}}{(z-1)^4} = \frac{1+e^2e^{2(z-1)}}{(z-1)^4}, \quad \text{for } z \neq 1.$$

Hence

$$f(z) = \frac{1}{(z-1)^4}\left(1+e^2\left(1+2(z-1)+\frac{4(z-1)^2}{2!} + \frac{8(z-1)^3}{3!}+\cdots\right)\right),$$

so

$$\operatorname{Res}(f,1) = \frac{8e^2}{3!} = \frac{4}{3}e^2.$$

Remark: We could have used the substitution $w = z-1$ to simplify the algebra slightly.

Solution to Exercise 1.8

(a) Since the function f has a pole of order two at the point π, we apply Theorem 1.3 with $k=2$. We obtain

$$\begin{aligned}\operatorname{Res}(f,\pi) &= \lim_{z\to\pi}\left(\frac{d}{dz}\left(ze^{iz}\right)\right)\\ &= \lim_{z\to\pi}\left(e^{iz}+ize^{iz}\right)\\ &= e^{i\pi}+i\pi e^{i\pi} = -1-i\pi.\end{aligned}$$

(Note that this answer agrees with that of Exercise 1.2(b).)

(b) Since the function f has a pole of order three at the point 0, we apply Theorem 1.3 with $k=3$. We obtain

$$\begin{aligned}\operatorname{Res}(f,0) &= \frac{1}{2!}\lim_{z\to 0}\left(\frac{d^2}{dz^2}\left(\frac{z+2}{z+4}\right)\right)\\ &= \frac{1}{2}\lim_{z\to 0}\left(\frac{d}{dz}\left(\frac{2}{(z+4)^2}\right)\right)\\ &= \frac{1}{2}\lim_{z\to 0}\left(\frac{-4}{(z+4)^3}\right)\\ &= \frac{1}{2}\left(\frac{-4}{4^3}\right) = -\frac{1}{32}.\end{aligned}$$

(Note that this answer agrees with that of Exercise 1.7(a).)

Solution to Exercise 1.9

If f has a pole of order k at α, then its Laurent series about α is

$$f(z) = \frac{a_{-k}}{(z-\alpha)^k}+\cdots+\frac{a_{-1}}{z-\alpha} + a_0 + a_1(z-\alpha)+\cdots,$$

where $a_{-k}\neq 0$, so

$$(z-\alpha)^k f(z) = a_{-k}+\cdots+a_{-1}(z-\alpha)^{k-1} + a_0(z-\alpha)^k + a_1(z-\alpha)^{k+1}+\cdots.$$

Using the Differentiation Rule for power series (Theorem 2.3 of Unit B3), we can differentiate $k-1$ times to obtain

$$\frac{d^{k-1}}{dz^{k-1}}\left((z-\alpha)^k f(z)\right) = (k-1)!\,a_{-1}+k!\,a_0(z-\alpha) + \frac{(k+1)!}{2!}a_1(z-\alpha)^2+\cdots.$$

Dividing by $(k-1)!$ and taking the limit as z tends to α gives

$$\frac{1}{(k-1)!}\lim_{z\to\alpha}\left(\frac{d^{k-1}}{dz^{k-1}}\left((z-\alpha)^k f(z)\right)\right) = a_{-1},$$

as required.

Solution to Exercise 1.10

(a) The Laurent series about 0 for the function f is

$$\begin{aligned}\frac{e^z}{z^7} &= \frac{1}{z^7}\left(1+z+\frac{z^2}{2!}+\cdots\right)\\ &= \frac{1}{z^7}+\cdots+\frac{1}{5!\,z^2}+\frac{1}{6!\,z}+\frac{1}{7!}+\cdots,\end{aligned}$$

so $\operatorname{Res}(f,0) = 1/6! = 1/720$.

Alternatively, applying Theorem 1.3 with $k=7$, we obtain

$$\begin{aligned}\operatorname{Res}(f,0) &= \frac{1}{6!}\lim_{z\to 0}\left(\frac{d^6}{dz^6}(e^z)\right)\\ &= \frac{1}{6!}\lim_{z\to 0}e^z = \frac{1}{6!}.\end{aligned}$$

(b) Substituting $w = z-\pi/2$, so $z = \pi/2+w$, we obtain, for $z\neq\pi/2$,

$$\begin{aligned}f(z) &= \frac{\cos z}{(z-\pi/2)^2}\\ &= \frac{\cos(\pi/2+w)}{w^2}\\ &= -\frac{\sin w}{w^2}\\ &= -\frac{1}{w^2}\left(w-\frac{w^3}{3!}+\cdots\right)\\ &= -\frac{1}{w}+\frac{w}{3!}-\cdots\\ &= -\frac{1}{z-\pi/2}+\frac{z-\pi/2}{3!}-\cdots.\end{aligned}$$

Hence $\operatorname{Res}(f,\pi/2) = -1$.

(c) We use the g/h Rule with $g(z) = e^z$, $h(z) = z^4 - 1$ and $h'(z) = 4z^3$. (The functions g and h are entire, and $h(\alpha) = 0$ and $h'(\alpha) \neq 0$ for $\alpha = 1, -1, i, -i$.) Thus

$$\operatorname{Res}(f,1) = \frac{g(1)}{h'(1)} = \frac{e^1}{4 \times 1^3} = \frac{e}{4},$$

$$\operatorname{Res}(f,-1) = \frac{g(-1)}{h'(-1)} = \frac{e^{-1}}{4 \times (-1)^3} = -\frac{e^{-1}}{4},$$

$$\operatorname{Res}(f,i) = \frac{g(i)}{h'(i)} = \frac{e^i}{4i^3} = \frac{ie^i}{4},$$

$$\operatorname{Res}(f,-i) = \frac{g(-i)}{h'(-i)} = \frac{e^{-i}}{4 \times (-i)^3} = -\frac{ie^{-i}}{4}.$$

(d) The function

$$f(z) = \frac{1}{z^2 - 4} = \frac{1}{(z-2)(z+2)}$$

has simple poles at 2 and -2. Applying the Cover-up Rule gives

$$\operatorname{Res}(f,2) = \frac{1}{2+2} = \frac{1}{4},$$

$$\operatorname{Res}(f,-2) = \frac{1}{-2-2} = -\frac{1}{4}.$$

(e) The function

$$f(z) = \frac{e^z}{z^3(z^2-9)} = \frac{e^z}{z^3(z-3)(z+3)}$$

has simple poles at 3 and -3. Applying the Cover-up Rule gives

$$\operatorname{Res}(f,3) = \frac{e^3}{3^3(3+3)} = \frac{e^3}{162},$$

$$\operatorname{Res}(f,-3) = \frac{e^{-3}}{(-3)^3(-3-3)} = \frac{e^{-3}}{162}.$$

Next, f has a pole of order three at 0. We will find $\operatorname{Res}(f,0)$ from the Laurent series about 0 for f. Observe that

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots, \quad \text{for } z \in \mathbb{C},$$

and

$$\frac{1}{z^2-9} = -\frac{1}{9} \times \frac{1}{1 - z^2/9} = -\frac{1}{9}\left(1 + \frac{z^2}{9} + \cdots\right),$$

for $|z| < 3$. Hence

$$f(z) = -\frac{1}{9z^3}\left(1 + z + \frac{z^2}{2!} + \cdots\right)\left(1 + \frac{z^2}{9} + \cdots\right),$$

for $|z| < 3$.

Since we need only find the coefficient of z^{-1} in this Laurent series, we determine the coefficient of z^2 in the product of the two brackets. This coefficient is

$$\frac{1}{9} + \frac{1}{2!} = \frac{11}{18},$$

so

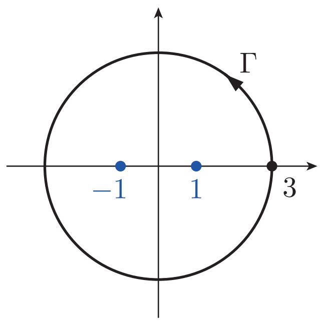
$$\operatorname{Res}(f,0) = -\frac{1}{9} \times \frac{11}{18} = -\frac{11}{162}.$$

(It is also possible to use Theorem 1.3 with $k = 3$ to find $\operatorname{Res}(f,0)$.)

Solution to Exercise 2.1

Let $f(z) = \dfrac{\sin z}{z^2 - 1} = \dfrac{\sin z}{(z-1)(z+1)}$.

(a) The function f is analytic on $\mathbb{C}$ apart from simple poles at 1 and -1, both of which lie inside the circle $\Gamma = \{z : |z| = 3\}$, as shown in the figure.



Using the Cover-up Rule, we obtain

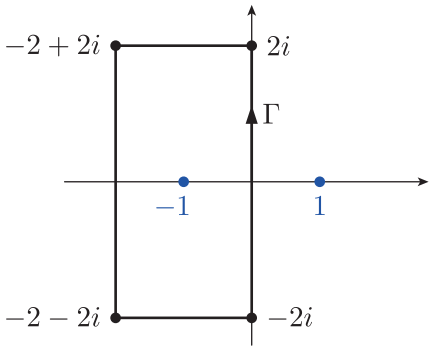
$$\operatorname{Res}(f,1) = \frac{\sin 1}{1+1} = \tfrac{1}{2}\sin 1,$$

$$\operatorname{Res}(f,-1) = \frac{\sin(-1)}{(-1)-1} = \tfrac{1}{2}\sin 1.$$

Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$,

$$\begin{aligned}\int_\Gamma \frac{\sin z}{z^2-1}\,dz &= 2\pi i(\operatorname{Res}(f,1) + \operatorname{Res}(f,-1)) \\ &= 2\pi i\left(\tfrac{1}{2}\sin 1 + \tfrac{1}{2}\sin 1\right) \\ &= 2\pi i \sin 1.\end{aligned}$$

(b) In this case the simple pole at -1 lies inside the rectangular contour Γ with vertices at $-2i$, $2i$, $-2+2i$, $-2-2i$, and the simple pole at 1 lies outside Γ, as shown in the following figure.



It follows from the Residue Theorem with $\mathcal{R} = \mathbb{C}$ that

$$\int_\Gamma \frac{\sin z}{z^2-1}\,dz = 2\pi i \operatorname{Res}(f,-1)$$
$$= 2\pi i \times \tfrac{1}{2}\sin 1 = \pi i \sin 1.$$

Solution to Exercise 2.2

The function

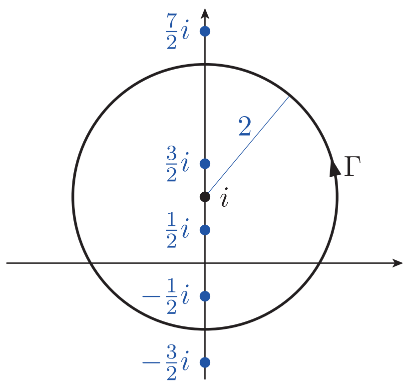
$$f(z) = \frac{z+2}{4z^2+k^2} = \frac{z+2}{4\left(z-\frac{1}{2}ki\right)\left(z+\frac{1}{2}ki\right)}$$

is analytic on $\mathbb{C}$ apart from simple poles at $\frac{1}{2}ki$ and $-\frac{1}{2}ki$. By the Cover-up Rule,

$$\operatorname{Res}\left(f,\tfrac{1}{2}ki\right) = \frac{\frac{1}{2}ki+2}{4ki} = \frac{ki+4}{8ki},$$

$$\operatorname{Res}\left(f,-\tfrac{1}{2}ki\right) = \frac{-\frac{1}{2}ki+2}{-4ki} = \frac{ki-4}{8ki}.$$

The figure shows the circle $\Gamma = \{z : |z-i| = 2\}$.



(a) If $k=1$, then the poles at $\frac{1}{2}i$ and $-\frac{1}{2}i$ both lie inside Γ. Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$,

$$I = 2\pi i\left(\operatorname{Res}\left(f,\tfrac{1}{2}i\right) + \operatorname{Res}\left(f,-\tfrac{1}{2}i\right)\right)$$
$$= 2\pi i\left(\frac{i+4}{8i} + \frac{i-4}{8i}\right)$$
$$= \frac{\pi i}{2}.$$

(b) If $k=3$, then the pole at $\frac{3}{2}i$ lies inside Γ and that at $-\frac{3}{2}i$ lies outside Γ. Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$,

$$I = 2\pi i \operatorname{Res}\left(f,\tfrac{3}{2}i\right)$$
$$= 2\pi i\left(\frac{3i+4}{24i}\right)$$
$$= \frac{\pi}{3} + \frac{\pi i}{4}.$$

(c) If $k=7$, then the poles at $\frac{7}{2}i$ and $-\frac{7}{2}i$ both lie outside Γ, so $I = 0$.

Remark: In this case the fact that the integral has value 0 also follows from Cauchy's Theorem, because f is analytic on the simply connected region $\mathcal{R} = \{z : -7/2 < \operatorname{Im} z < 7/2\}$, which contains Γ.

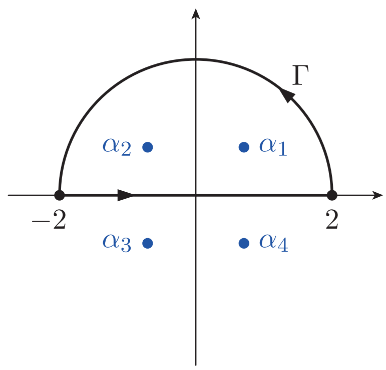
Solution to Exercise 2.3

The function

$$f(z) = \frac{z^3}{z^4+1}$$

is analytic on $\mathbb{C}$ apart from simple poles at the points where the denominator is zero, that is, at $\alpha_1 = e^{\pi i/4}$, $\alpha_2 = e^{3\pi i/4}$, $\alpha_3 = e^{5\pi i/4}$ and $\alpha_4 = e^{7\pi i/4}$.

Of these poles, the first two lie inside Γ and the second two lie outside Γ (see the figure). The residue at each of these poles is $\frac{1}{4}$ (see Exercise 1.4(c)).

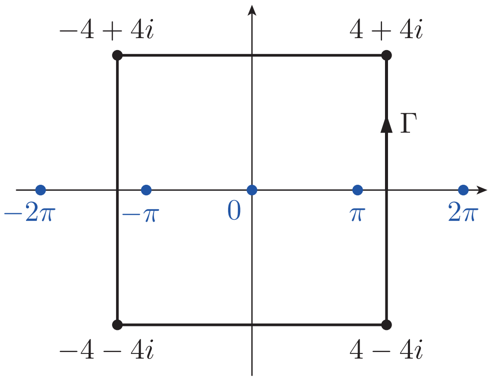


Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$,

$$\int_\Gamma \frac{z^3}{z^4+1}\,dz = 2\pi i\left(\operatorname{Res}\left(f,e^{\pi i/4}\right) + \operatorname{Res}\left(f,e^{3\pi i/4}\right)\right)$$
$$= 2\pi i\left(\tfrac{1}{4}+\tfrac{1}{4}\right) = \pi i.$$

Solution to Exercise 2.4

The function $f(z) = (1+z)/\sin z$ is analytic on $\mathbb{C}$ apart from simple poles at the points where $\sin z = 0$, that is, at $z = n\pi$, where n is an integer. (So f has infinitely many simple poles.) Of these poles, only those at $0, \pi$ and $-\pi$ lie inside the given square contour Γ (see the figure). None of the poles lies on Γ.



Let $g(z) = 1+z$ and $h(z) = \sin z$, so $h'(z) = \cos z$. Then g and h are analytic at α, for $\alpha = 0, \pi, -\pi$, and $h(\alpha) = 0$ but $h'(\alpha) \neq 0$. By the g/h Rule,

$$\operatorname{Res}(f, 0) = (1+0)/\cos 0 = 1,$$
$$\operatorname{Res}(f, \pi) = (1+\pi)/\cos \pi = -1-\pi,$$
$$\operatorname{Res}(f, -\pi) = (1-\pi)/\cos(-\pi) = -1+\pi.$$

Hence, by the Residue Theorem with, for example, $\mathcal{R} = \{z : |\operatorname{Re} z| < 5\}$, we have

$$\begin{aligned}
&\int_\Gamma \frac{1+z}{\sin z}\,dz \\
&= 2\pi i(\operatorname{Res}(f,0) + \operatorname{Res}(f,\pi) + \operatorname{Res}(f,-\pi)) \\
&= 2\pi i(1 + (-1-\pi) + (-1+\pi)) \\
&= -2\pi i.
\end{aligned}$$

(Note that the choice of $\mathcal{R} = \mathbb{C}$ is not acceptable since f has infinitely many poles in $\mathbb{C}$.)

Solution to Exercise 2.5

Using the strategy for evaluating real trigonometric integrals, we obtain

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{2-\cos t}\,dt &= \int_C \frac{1}{2-(z+z^{-1})/2} \times \frac{1}{iz}\,dz \\
&= \frac{1}{i}\int_C \frac{1}{2z-(z^2+1)/2}\,dz \\
&= 2i\int_C \frac{1}{z^2-4z+1}\,dz,
\end{aligned}$$

where $C = \{z : |z| = 1\}$.

The function

$$f(z) = \frac{1}{z^2-4z+1} = \frac{1}{(z-(2-\sqrt{3}))(z-(2+\sqrt{3}))}$$

is analytic on $\mathbb{C}$ except for simple poles at $2 \pm \sqrt{3}$. The pole $2-\sqrt{3}$ lies inside C, and the pole $2+\sqrt{3}$ lies outside C.

By the Cover-up Rule,

$$\operatorname{Res}(f, 2-\sqrt{3}) = \frac{1}{(2-\sqrt{3})-(2+\sqrt{3})} = -\frac{1}{2\sqrt{3}}.$$

Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$,

$$\int_C \frac{1}{z^2-4z+1}\,dz = 2\pi i \times \left(-\frac{1}{2\sqrt{3}}\right) = -\frac{\pi i}{\sqrt{3}}.$$

Therefore

$$\int_0^{2\pi} \frac{1}{2-\cos t}\,dt = 2i \times \left(-\frac{\pi i}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}.$$

Solution to Exercise 2.6

(a) The residue of the function

$$f(z) = \frac{(z^2+1)^n}{z^{n+1}}$$

at the point 0 is the coefficient of z^{-1} in the expansion of $(z^2+1)^n/z^{n+1}$; this is just the coefficient of z^n in the expansion of $(z^2+1)^n$, which is

$$\binom{n}{\frac{1}{2}n} \text{ if } n \text{ is even, and 0 if } n \text{ is odd.}$$

(b) Using the strategy for evaluating real trigonometric integrals, we obtain

$$\begin{aligned}
\int_0^{2\pi} \cos^n t\,dt &= \int_C \left(\frac{1}{2}(z+z^{-1})\right)^n \frac{1}{iz}\,dz \\
&= \frac{1}{2^n i}\int_C \frac{(z^2+1)^n}{z^{n+1}}\,dz.
\end{aligned}$$

The only singularity of f is a pole of order $n+1$ at the point 0 (which lies inside the unit circle C), and the residue of f at 0 is as given in part (a). Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$, the value of the original integral is

$$\frac{1}{2^n i} \times 2\pi i \operatorname{Res}(f, 0) = \begin{cases} \dfrac{\pi}{2^{n-1}}\dbinom{n}{\frac{1}{2}n}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Solution to Exercise 2.7

(a) The function $f(z) = (\cos z)/(z - \pi/2)^2$ is analytic on $\mathbb{C}$ apart from a singularity at $\pi/2$, which lies inside $C = \{z : |z| = 2\}$. Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$ and Exercise 1.10(b),

$$\int_C \frac{\cos z}{(z - \pi/2)^2}\, dz = 2\pi i(-1) = -2\pi i.$$

(b) The function $f(z) = e^z/(z^4 - 1)$ is analytic on $\mathbb{C}$ apart from singularities at 1, -1, i and $-i$, all of which lie inside $C = \{z : |z| = 2\}$. Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$ and Exercise 1.10(c),

$$\begin{aligned}\int_C \frac{e^z}{z^4 - 1}\, dz &= 2\pi i\left(\tfrac{1}{4}e - \tfrac{1}{4}e^{-1} + \tfrac{1}{4}ie^{i} - \tfrac{1}{4}ie^{-i}\right)\\ &= \pi i(\sinh 1 - \sin 1).\end{aligned}$$

(c) The function $f(z) = e^z/\big(z^3(z^2 - 9)\big)$ is analytic on $\mathbb{C}$ apart from singularities at 0, 3 and -3. Of these, 0 lies inside $C = \{z : |z| = 2\}$, and 3 and -3 lie outside C. Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$ and Exercise 1.10(e),

$$\int_C \frac{e^z}{z^3(z^2 - 9)}\, dz = 2\pi i\left(-\frac{11}{162}\right) = -\frac{11\pi i}{81}.$$

Solution to Exercise 2.8

The function $f(z) = z/(e^z - 1)$ is analytic on $\mathbb{C}$ apart from singularities at the points $2k\pi i$, for $k \in \mathbb{Z}$ (the zeros of $e^z - 1$).

Using the g/h Rule with $g(z) = z$, $h(z) = e^z - 1$ and $h'(z) = e^z$ (where $h(2k\pi i) = 0$ and $h'(2k\pi i) = 1 \neq 0$), we obtain

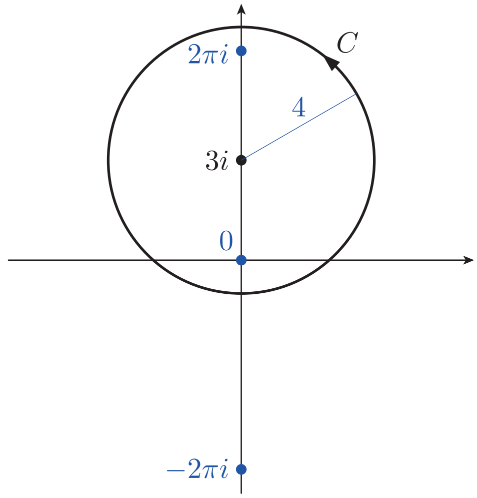
$$\operatorname{Res}(f, 2k\pi i) = \frac{2k\pi i}{e^{2k\pi i}} = 2k\pi i, \quad \text{for } k \in \mathbb{Z}.$$

(a) The only singularity of f inside $C = \{z : |z| = 1\}$ is 0; the others lie outside C. Hence, by the Residue Theorem with $\mathcal{R} = \{z : |z| < 2\}$,

$$\int_C \frac{z}{e^z - 1}\, dz = 2\pi i \operatorname{Res}(f, 0) = 0.$$

(This result is as expected since f has a removable singularity at 0 which, when removed, makes f analytic on $\mathcal{R}$, so Cauchy's Theorem can then be applied.)

(b) The singularities of f inside $C = \{z : |z - 3i| = 4\}$ are 0 and $2\pi i$; the others lie outside C.



Hence, by the Residue Theorem with $\mathcal{R} = \{z : |z - 3i| < 5\}$,

$$\begin{aligned}\int_C \frac{z}{e^z - 1}\, dz &= 2\pi i(\operatorname{Res}(f, 0) + \operatorname{Res}(f, 2\pi i))\\ &= 2\pi i(0 + 2\pi i) = -4\pi^2.\end{aligned}$$

Solution to Exercise 2.9

Using the strategy for evaluating real trigonometric integrals, we obtain

$$\begin{aligned}&\int_0^{2\pi} \frac{1}{16\cos^2 t + 9}\, dt\\ &= \int_C \frac{1}{16 \times \frac{1}{4}(z + z^{-1})^2 + 9} \times \frac{1}{iz}\, dz\\ &= \int_C \frac{1}{4(z^2 + 2 + z^{-2}) + 9} \times \frac{1}{iz}\, dz\\ &= \int_C \frac{1}{iz(4z^2 + 17 + 4z^{-2})}\, dz\\ &= \int_C \frac{z}{i(4z^4 + 17z^2 + 4)}\, dz\\ &= \frac{1}{4i}\int_C \frac{z}{(z^2 + 4)\left(z^2 + \frac{1}{4}\right)}\, dz,\end{aligned}$$

as required.

Now, the singularities of the function $f(z) = z/\big((z^2 + 4)\left(z^2 + \frac{1}{4}\right)\big)$ are simple poles at $2i$, $-2i$, $\frac{1}{2}i$ and $-\frac{1}{2}i$. Of these poles, only the last two lie inside the unit circle C; the other two lie outside C.

Using the g/h Rule with $g(z) = z/(z^2+4)$, $h(z) = z^2 + \frac{1}{4}$ and $h'(z) = 2z$, where $h(\frac{1}{2}i) = 0$ and $h'(\frac{1}{2}i) = i \neq 0$, we obtain

$$\operatorname{Res}(f, \tfrac{1}{2}i) = \left(\frac{\frac{1}{2}i}{(\frac{1}{2}i)^2 + 4}\right) \Big/ \left(2 \times (\tfrac{1}{2}i)\right) = \frac{2}{15}.$$

Since f is an odd function, we can apply Theorem 1.1(a) to give

$$\operatorname{Res}(f, -\tfrac{1}{2}i) = \operatorname{Res}(f, \tfrac{1}{2}i) = \frac{2}{15}.$$

Hence, by the Residue Theorem with $\mathcal{R} = \mathbb{C}$, the value of the original integral is

$$\begin{aligned}&\frac{1}{4i} \times 2\pi i\left(\operatorname{Res}(f, \tfrac{1}{2}i) + \operatorname{Res}(f, -\tfrac{1}{2}i)\right)\\ &= \frac{\pi}{2}\left(\frac{2}{15} + \frac{2}{15}\right) = \frac{2\pi}{15}.\end{aligned}$$

Solution to Exercise 3.1

Let $f(r) = 1/\sqrt{r}$ $(r \in (0, \infty))$. Choose any positive number ε. Then, for $r > 0$,

$$\begin{aligned}\left|\frac{1}{\sqrt{r}} - 0\right| < \varepsilon &\iff \frac{1}{\sqrt{r}} < \varepsilon\\ &\iff \frac{1}{r} < \varepsilon^2\\ &\iff r > 1/\varepsilon^2.\end{aligned}$$

Let N be any positive integer greater than $1/\varepsilon^2$. Then

$$\left|\frac{1}{\sqrt{r}} - 0\right| < \varepsilon, \quad \text{for all } r > N.$$

Thus $\lim\limits_{r\to\infty} \dfrac{1}{\sqrt{r}} = 0$.

Solution to Exercise 3.2

Let

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

where $a_n \neq 0$, and

$$q(z) = b_0 + b_1 z + \cdots + b_m z^m,$$

where $b_m \neq 0$. Suppose that the degree m of q exceeds the degree n of p, that is, $m > n$. Then

$$\begin{aligned}&\lim_{r\to\infty} \frac{p(r)}{q(r)}\\ &= \lim_{r\to\infty} \frac{p(r)/r^m}{q(r)/r^m}\\ &= \lim_{r\to\infty} \frac{a_0 r^{-m} + a_1 r^{-m+1} + \cdots + a_n r^{-m+n}}{b_0 r^{-m} + b_1 r^{-m+1} + \cdots + b_{m-1} r^{-1} + b_m}.\end{aligned}$$

Since $\lim\limits_{r\to\infty} 1/r = 0$ and $m > n$, it follows from the Combination Rules for limits of functions (Theorem 3.1) that

$$\lim_{r\to\infty} \frac{p(r)}{q(r)} = \frac{0 + 0 + \cdots + 0}{0 + 0 + \cdots + 0 + b_m} = 0.$$

Solution to Exercise 3.3

(a)
$$\begin{aligned}\int_{-\infty}^{\infty} \sin t\, dt &= \lim_{r\to\infty} \int_{-r}^{r} \sin t\, dt\\ &= \lim_{r\to\infty} [-\cos t]_{-r}^{r} = \lim_{r\to\infty} 0 = 0\end{aligned}$$

(b)
$$\begin{aligned}\int_{1}^{\infty} \frac{1}{t^p}\, dt &= \lim_{r\to\infty} \int_{1}^{r} \frac{1}{t^p}\, dt\\ &= \lim_{r\to\infty} \left[\frac{1}{1-p} \times \frac{1}{t^{p-1}}\right]_1^r \quad (\text{as } p > 1)\\ &= \lim_{r\to\infty} \frac{1}{1-p}\left(\frac{1}{r^{p-1}} - 1\right)\\ &= 1/(p-1)\end{aligned}$$

(c)
$$\begin{aligned}\int_{0}^{\infty} e^{-t}\, dt &= \lim_{r\to\infty} \int_{0}^{r} e^{-t}\, dt\\ &= \lim_{r\to\infty} \left[-e^{-t}\right]_0^r\\ &= \lim_{r\to\infty} \left(-e^{-r} - (-1)\right) = 1\end{aligned}$$

Solution to Exercise 3.4

Suppose that f is an even function and both the improper integrals

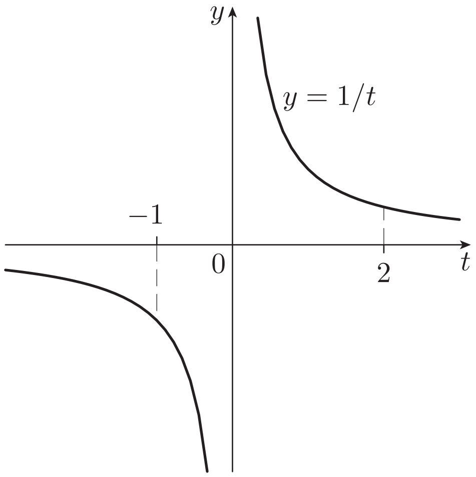
$$\int_{-\infty}^{\infty} f(t)\, dt \quad \text{and} \quad \int_{0}^{\infty} f(t)\, dt$$

exist. Then

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\, dt &= \lim_{r\to\infty} \int_{-r}^{r} f(t)\, dt\\ &= \lim_{r\to\infty} \left(\int_0^r f(t)\, dt + \int_{-r}^{0} f(t)\, dt\right)\\ &= \lim_{r\to\infty} \left(\int_0^r f(t)\, dt + \int_{r}^{0} f(-u)(-du)\right)\\ &= \lim_{r\to\infty} \left(\int_0^r f(t)\, dt + \int_{0}^{r} f(u)\, du\right)\\ &= 2\lim_{r\to\infty} \int_0^r f(t)\, dt\\ &= 2\int_0^{\infty} f(t)\, dt.\end{aligned}$$

Solution to Exercise 3.5

The function $f(t) = 1/t$ is continuous at all points of $[-1, 2]$ except at the point 0, as shown in the figure.



Recall that the function $F(t) = \log|t|$ $(t \in \mathbb{R} - \{0\})$ satisfies $F'(t) = 1/t$, for $t \neq 0$. So

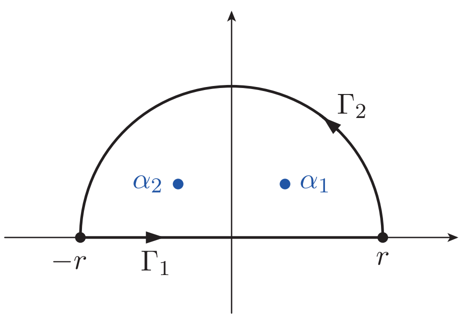
$$\begin{aligned}\int_{-1}^{2} \frac{1}{t}\,dt &= \lim_{\varepsilon\to 0}\left(\int_{-1}^{-\varepsilon} \frac{1}{t}\,dt + \int_{\varepsilon}^{2} \frac{1}{t}\,dt\right)\\ &= \lim_{\varepsilon\to 0}\left(\big[\log|t|\big]_{-1}^{-\varepsilon} + \big[\log|t|\big]_{\varepsilon}^{2}\right)\\ &= \lim_{\varepsilon\to 0}\big((\log\varepsilon - \log 1) + (\log 2 - \log\varepsilon)\big)\\ &= \lim_{\varepsilon\to 0}(\log 2 - \log 1)\\ &= \log 2.\end{aligned}$$

Solution to Exercise 3.6

1. Consider the contour integral

$$I = \int_{\Gamma} \frac{1}{z^4+1}\,dz,$$

where $\Gamma = \Gamma_1 + \Gamma_2$ is the contour shown in the figure (with $r > 1$).



The function $f(z) = 1/(z^4+1)$ is analytic on $\mathbb{C}$ apart from simple poles at $\alpha_1 = e^{\pi i/4}$, $\alpha_2 = e^{3\pi i/4}$, $\alpha_3 = e^{5\pi i/4}$, $\alpha_4 = e^{7\pi i/4}$.

Since $r > 1$, the poles at $\alpha_1 = e^{\pi i/4}$ and $\alpha_2 = e^{3\pi i/4}$ lie inside Γ; the other two lie outside Γ.

2. By the Residue Theorem,

$$\begin{aligned}I &= 2\pi i\Big(\operatorname{Res}(f, e^{\pi i/4}) + \operatorname{Res}(f, e^{3\pi i/4})\Big)\\ &= 2\pi i\left(\frac{1}{4}e^{-3\pi i/4} + \frac{1}{4}e^{-9\pi i/4}\right) \quad \text{(by the } g/h \text{ Rule)}\\ &= \frac{\pi i}{2}\left(\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) + \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right)\\ &= \pi/\sqrt{2}.\end{aligned}$$

3. Splitting up the integral gives

$$\begin{aligned}I &= \int_{\Gamma_1} \frac{1}{z^4+1}\,dz + \int_{\Gamma_2} \frac{1}{z^4+1}\,dz\\ &= \int_{-r}^{r} \frac{1}{t^4+1}\,dt + \int_{\Gamma_2} \frac{1}{z^4+1}\,dz. \qquad \text{(S1)}\end{aligned}$$

4. By the backwards form of the Triangle Inequality,

$$|z^4+1| \geq |z^4| - 1 = |z|^4 - 1 = r^4 - 1,$$

for $z \in \Gamma_2$. Since f is continuous on Γ_2, and the length of Γ_2 is πr, we see from the Estimation Theorem that

$$\left|\int_{\Gamma_2} \frac{1}{z^4+1}\,dz\right| \leq \frac{1}{r^4-1} \times \pi r = \frac{\pi r}{r^4-1},$$

for $r > 1$.

5. From step 4,

$$\lim_{r\to\infty} \int_{\Gamma_2} \frac{1}{z^4+1}\,dz = 0,$$

since $\pi r/(r^4-1) \to 0$ as $r \to \infty$, by the corollary to Theorem 3.1. Hence taking the limit of each expression in equation (S1) as $r \to \infty$ gives

$$\pi/\sqrt{2} = \lim_{r\to\infty} \int_{-r}^{r} \frac{1}{t^4+1}\,dt + 0.$$

So this limit exists, and has value

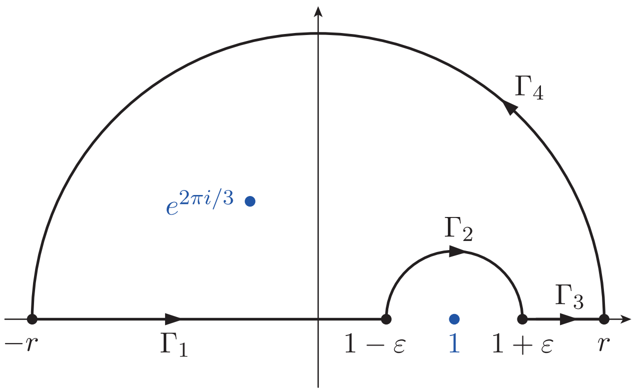
$$\int_{-\infty}^{\infty} \frac{1}{t^4+1}\,dt = \pi/\sqrt{2}.$$

Solution to Exercise 3.7

1. Consider the contour integral

$$I = \int_\Gamma \frac{z}{z^3 - 1}\, dz,$$

where $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ is the contour shown in the figure (with $0 < \varepsilon < 1$ and $r > 2$).



The function $f(z) = z/(z^3 - 1)$ is analytic on $\mathbb{C}$ apart from simple poles at 1, $e^{2\pi i/3}$ and $e^{4\pi i/3}$. The pole at $e^{2\pi i/3}$ lies inside Γ; the other two poles lie outside Γ.

2. By the Residue Theorem,

$$\begin{aligned} I &= 2\pi i \operatorname{Res}(f, e^{2\pi i/3}) \\ &= 2\pi i \times \frac{e^{2\pi i/3}}{3e^{4\pi i/3}} \quad \text{(by the } g/h \text{ Rule)} \\ &= \frac{2\pi i}{3} e^{-2\pi i/3} \\ &= \frac{\pi i}{3}(-1 - i\sqrt{3}). \end{aligned}$$

3. Splitting up the integral gives

$$\begin{aligned} I = &\int_{-r}^{1-\varepsilon} \frac{t}{t^3 - 1}\, dt + \int_{1+\varepsilon}^{r} \frac{t}{t^3-1}\, dt \\ &+ \int_{\Gamma_2} \frac{z}{z^3-1}\, dz + \int_{\Gamma_4} \frac{z}{z^3-1}\, dz. \qquad \text{(S2)} \end{aligned}$$

4. By the backwards form of the Triangle Inequality and the Estimation Theorem, we have

$$\left| \int_{\Gamma_4} \frac{z}{z^3-1}\, dz \right| \le \frac{r}{r^3 - 1} \times \pi r = \frac{\pi r^2}{r^3 - 1},$$

for $r > 2$.

5. By step 4,

$$\lim_{r\to\infty} \int_{\Gamma_4} \frac{z}{z^3-1}\, dz = 0.$$

Also, using the Round-the-Pole Lemma,

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\Gamma_2} \frac{z}{z^3-1}\, dz &= -\pi i \operatorname{Res}(f, 1) \\ &= -\pi i \times 1/3 \quad \text{(by the } g/h \text{ Rule)} \\ &= -\pi i/3. \end{aligned}$$

Therefore taking the limit as $\varepsilon \to 0$ and then the limit as $r \to \infty$ of each expression in equation (S2), we obtain

$$\frac{\pi i}{3}(-1 - i\sqrt{3}) = \int_{-\infty}^{\infty} \frac{t}{t^3-1}\, dt - \frac{\pi i}{3} + 0,$$

using the value for I found in step 2. Hence

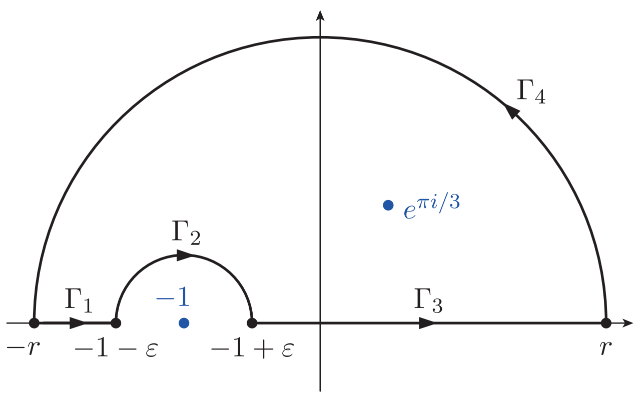
$$\int_{-\infty}^{\infty} \frac{t}{t^3 - 1}\, dt = \frac{\pi\sqrt{3}}{3} = \frac{\pi}{\sqrt{3}}.$$

Solution to Exercise 3.8

1. Consider the contour integral

$$I = \int_\Gamma \frac{e^{2iz}}{z^3 + 1}\, dz,$$

where $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ is the contour shown in the figure (with $0 < \varepsilon < 1$ and $r > 2$).



The function $f(z) = e^{2iz}/(z^3 + 1)$ is analytic on $\mathbb{C}$ apart from simple poles at -1, $e^{\pi i/3}$ and $e^{-\pi i/3}$. The pole at $e^{\pi i/3}$ lies inside Γ, for $r > 2$; the other two poles lie outside Γ.

2. By the Residue Theorem,

$$\begin{aligned} I &= 2\pi i\,\mathrm{Res}(f, e^{\pi i/3}) \\ &= 2\pi i \times \frac{\exp(2ie^{\pi i/3})}{3e^{2\pi i/3}} \quad \text{(by the } g/h \text{ Rule)} \\ &= \frac{2\pi i}{3}e^{-2\pi i/3} \times \exp(-\sqrt{3}+i) \\ &= \frac{\pi}{3}(-i+\sqrt{3})e^{-\sqrt{3}+i} \\ &= \frac{\pi}{3}e^{-\sqrt{3}}(\sqrt{3}-i)e^{i}. \end{aligned}$$

3. Splitting up the integrals gives

$$\begin{aligned} I &= \int_{-r}^{-1-\varepsilon} \frac{e^{2it}}{t^3+1}\,dt + \int_{-1+\varepsilon}^{r} \frac{e^{2it}}{t^3+1}\,dt \\ &\quad + \int_{\Gamma_2} \frac{e^{2iz}}{z^3+1}\,dz + \int_{\Gamma_4} \frac{e^{2iz}}{z^3+1}\,dz \\ &= \int_{-r}^{-1-\varepsilon} \frac{\cos 2t}{t^3+1}\,dt + \int_{-1+\varepsilon}^{r} \frac{\cos 2t}{t^3+1}\,dt \\ &\quad + i\int_{-r}^{-1-\varepsilon} \frac{\sin 2t}{t^3+1}\,dt + i\int_{-1+\varepsilon}^{r} \frac{\sin 2t}{t^3+1}\,dt \\ &\quad + \int_{\Gamma_2} \frac{e^{2iz}}{z^3+1}\,dz + \int_{\Gamma_4} \frac{e^{2iz}}{z^3+1}\,dz, \end{aligned} \tag{S3}$$

since $e^{2it} = \cos 2t + i\sin 2t$.

4. Observe that if $z = x+iy$, then

$$|e^{2iz}| = |e^{2ix}||e^{-2y}| = e^{-2y} \le 1, \quad \text{for } y \ge 0.$$

Then, using the backwards form of the Triangle Inequality and the Estimation Theorem, we obtain

$$\left|\int_{\Gamma_4} \frac{e^{2iz}}{z^3+1}\,dz\right| \le \frac{1}{r^3-1} \times \pi r = \frac{\pi r}{r^3-1},$$

for $r > 2$.

5. Using the Round-the-Pole Lemma, we have

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_{\Gamma_2} \frac{e^{2iz}}{z^3+1}\,dz &= -\pi i\,\mathrm{Res}(f,-1) \\ &= -\pi i \times \frac{e^{-2i}}{3\times(-1)^2} \\ &= -\frac{\pi}{3}\sin 2 - i\frac{\pi}{3}\cos 2, \end{aligned}$$

where we have used the g/h Rule to calculate the residue of f at -1. Also, by step 4,

$$\lim_{r\to\infty}\int_{\Gamma_4} \frac{e^{2iz}}{z^3+1}\,dz = 0.$$

Therefore taking the limit as $\varepsilon \to 0$ and then the limit as $r \to \infty$ of the expressions in equation (S3), we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\cos 2t}{t^3+1}\,dt + i\int_{-\infty}^{\infty} \frac{\sin 2t}{t^3+1}\,dt \\ &\quad - \frac{\pi}{3}\sin 2 - i\frac{\pi}{3}\cos 2 + 0 \\ &= \frac{\pi}{3}e^{-\sqrt{3}}(\sqrt{3}-i)e^{i}. \end{aligned}$$

On equating imaginary parts, we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\sin 2t}{t^3+1}\,dt \\ &= \frac{\pi}{3}\Big(\cos 2 + e^{-\sqrt{3}}(\sqrt{3}\sin 1 - \cos 1)\Big). \end{aligned}$$

Solution to Exercise 3.9

We apply Theorem 3.3 with $p(z) = 1$ and

$$q(z) = z(z-1)(z^2+1) = z(z-1)(z+i)(z-i).$$

To see that the conditions of the theorem are satisfied, first observe that the degree of q exceeds that of p by 4. Next, the only singularities of the function p/q are simple poles at 0, 1, $-i$ and i. Of these poles, i lies in the upper half-plane, and 0 and 1 lie on the real axis.

By Theorem 3.3,

$$\int_{-\infty}^{\infty} \frac{1}{t(t-1)(t^2+1)}\,dt = 2\pi i S + \pi i T,$$

where

$$\begin{aligned} S &= \mathrm{Res}(p/q, i) \\ &= \frac{1}{i(i-1)2i} \\ &= \frac{1}{2(1-i)} = \frac{1+i}{4}, \end{aligned}$$

by the Cover-up Rule, and

$$\begin{aligned} T &= \mathrm{Res}(p/q, 0) + \mathrm{Res}(p/q, 1) \\ &= \frac{1}{(-1)\times 1} + \frac{1}{1\times 2} = -\frac{1}{2}, \end{aligned}$$

by the Cover-up Rule. Thus

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{t(t-1)(t^2+1)}\,dt &= 2\pi i\left(\frac{1+i}{4}\right) - \frac{\pi i}{2} \\ &= -\frac{\pi}{2}. \end{aligned}$$

Solution to Exercise 3.10

We apply Theorem 3.3 with $p(z) = 1$ and

$$\begin{aligned} q(z) &= (z^2 + a^2)(z^2 + b^2) \\ &= (z + ai)(z - ai)(z + bi)(z - bi), \end{aligned}$$

where $a, b > 0$ and $a \neq b$. To see that the conditions of the theorem are satisfied, first observe that the degree of q exceeds that of p by 4. Next, the only singularities of the function p/q are simple poles at $-ai$, ai, $-bi$ and bi. Of these poles, only ai and bi lie in the upper half-plane. None of the poles lies on the real axis.

By Theorem 3.3,

$$\int_{-\infty}^{\infty} \frac{1}{(t^2 + a^2)(t^2 + b^2)}\, dt = 2\pi i S + \pi i T,$$

where $T = 0$ and

$$\begin{aligned} S &= \operatorname{Res}(p/q, ai) + \operatorname{Res}(p/q, bi) \\ &= \frac{1}{2ai(-a^2 + b^2)} + \frac{1}{(-b^2 + a^2)2bi}, \end{aligned}$$

by the Cover-up Rule. Thus

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{1}{(t^2 + a^2)(t^2 + b^2)}\, dt \\ &= 2\pi i \left(\frac{1}{2ai(b^2 - a^2)} + \frac{1}{2bi(a^2 - b^2)} \right) \\ &= \frac{\pi}{ab(b^2 - a^2)}(b - a) \\ &= \frac{\pi}{ab(a + b)}. \end{aligned}$$

Solution to Exercise 3.11

We apply Theorem 3.4 with $p(z) = 1$ and $q(z) = z$. To see that the conditions of the theorem are satisfied, first observe that the degree of q exceeds that of p by 1. Next, the only singularity of the function p/q is a simple pole at 0.

Furthermore, the only singularity of the function $f(z) = e^{ikz}/z$ is a simple pole at 0.

By Theorem 3.4,

$$\int_{-\infty}^{\infty} \frac{e^{ikt}}{t}\, dt = 2\pi i S + \pi i T,$$

where $S = 0$ and

$$T = \operatorname{Res}(f, 0) = \frac{e^0}{1} = 1,$$

by the Cover-up Rule. Thus

$$\int_{-\infty}^{\infty} \frac{e^{ikt}}{t}\, dt = i\pi.$$

Since $e^{ikt} = \cos kt + i \sin kt$, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos kt}{t}\, dt = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ikt}}{t}\, dt \right) = 0,$$

$$\int_{-\infty}^{\infty} \frac{\sin kt}{t}\, dt = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{e^{ikt}}{t}\, dt \right) = \pi.$$

Solution to Exercise 3.12

(a) We apply Theorem 3.3 with $p(z) = z^2$ and $q(z) = (z^2 + 4)^2$. To see that the conditions of the theorem are satisfied, observe that the degree of q exceeds that of p by 2, and p/q has poles at $-2i$ and $2i$, neither of which lies on the real axis.

The only pole of p/q that lies in the upper half-plane is $2i$, which is of order 2. We have

$$\begin{aligned} \operatorname{Res}(p/q, 2i) &= \lim_{z \to 2i} \frac{d}{dz} \left(\frac{(z - 2i)^2 z^2}{(z^2 + 4)^2} \right) \\ &= \lim_{z \to 2i} \frac{d}{dz} \left(\frac{z^2}{(z + 2i)^2} \right) \\ &= \lim_{z \to 2i} \frac{(z + 2i)^2 \times 2z - z^2 \times 2(z + 2i)}{(z + 2i)^4} \\ &= \lim_{z \to 2i} \frac{4iz}{(z + 2i)^3} \\ &= -\tfrac{1}{8} i, \end{aligned}$$

by Theorem 1.3. Hence, by Theorem 3.3,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{t^2}{(t^2 + 4)^2}\, dt &= 2\pi i S + \pi i T \\ &= 2\pi i \times \left(-\tfrac{1}{8} i\right) + \pi i \times 0 = \frac{\pi}{4}. \end{aligned}$$

(b) We apply Theorem 3.4 with $p(z) = 1$, $q(z) = z^2 + 9$ and $k = 1$. To see that the conditions of the theorem are satisfied, observe that the degree of q exceeds that of p by 2, and p/q has poles at $3i$ and $-3i$, neither of which lies on the real axis.

The only singularity of the function

$$f(z) = \frac{e^{iz}}{z^2 + 9} = \frac{e^{iz}}{(z - 3i)(z + 3i)}$$

in the upper half-plane is a simple pole at $3i$, and

$$\operatorname{Res}(f, 3i) = \frac{e^{i \times 3i}}{3i + 3i} = \frac{e^{-3}}{6i},$$

by the Cover-up Rule. Hence, by Theorem 3.4,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{it}}{t^2+9}\,dt &= 2\pi i S + \pi i T\\ &= 2\pi i \times \frac{e^{-3}}{6i} + \pi i \times 0 = \frac{\pi}{3e^3}.\end{aligned}$$

On equating real parts we obtain

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+9}\,dt = \frac{\pi}{3e^3}.$$

Solution to Exercise 3.13

(a) We have

$$\int_0^r \frac{1}{(t+1)^2}\,dt = \left[-\frac{1}{t+1}\right]_0^r = 1 - \frac{1}{r+1}.$$

Since $1/(r+1) \to 0$ as $r \to \infty$, we deduce that

$$\int_0^{\infty} \frac{1}{(t+1)^2}\,dt = \lim_{r\to\infty} \int_0^r \frac{1}{(t+1)^2}\,dt = 1.$$

(b) Since the integrand is an odd function that is continuous on $\mathbb{R}$, we deduce from Theorem 3.2(a) that

$$\int_{-\infty}^{\infty} \frac{t}{t^4+1}\,dt = 0.$$

(c) From Exercise 3.12(a), we know that

$$\int_{-\infty}^{\infty} \frac{t^2}{(t^2+4)^2}\,dt = \frac{\pi}{4}.$$

Since the integrand is an even function that is continuous on $\mathbb{R}$, we deduce from Theorem 3.2(b) that

$$\int_0^{\infty} \frac{t^2}{(t^2+4)^2}\,dt = \frac{1}{2}\int_{-\infty}^{\infty} \frac{t^2}{(t^2+4)^2}\,dt = \frac{\pi}{8}.$$

(d) We apply Theorem 3.3 with $p(z) = z$ and $q(z) = z^3 + 1$. To see that the conditions of the theorem are satisfied, observe that the degree of q exceeds that of p by 2, and p/q has poles at -1, $e^{\pi i/3}$ and $e^{-\pi i/3}$, all of which are simple.

Of these poles, only $e^{\pi i/3}$ lies in the upper half-plane, and only -1 lies on the real axis. Now, using the g/h Rule with $g(z) = p(z) = z$, $h(z) = q(z) = z^3 + 1$ and $h'(z) = 3z^2$, we obtain

$$\operatorname{Res}(f,-1) = \frac{-1}{3(-1)^2} = -\frac{1}{3},$$

$$\operatorname{Res}(f,e^{\pi i/3}) = \frac{e^{\pi i/3}}{3e^{2\pi i/3}} = \frac{e^{-\pi i/3}}{3}.$$

Hence, by Theorem 3.3,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{t}{t^3+1}\,dt &= 2\pi i S + \pi i T\\ &= 2\pi i \times \frac{e^{-\pi i/3}}{3} + \pi i \times \left(-\frac{1}{3}\right)\\ &= \frac{\pi i}{3}(2e^{-\pi i/3} - 1)\\ &= \frac{\pi i}{3}((1 - i\sqrt{3}) - 1) = \frac{\pi}{\sqrt{3}}.\end{aligned}$$

(e) We apply Theorem 3.4 with $p(z) = z$, $q(z) = 1 - z^2$ and $k = \pi$. To see that the conditions of the theorem are satisfied, observe that the degree of q exceeds that of p by 1, and p/q has poles at 1 and -1, both of which are simple.

The only singularities of the function

$$f(z) = \frac{p(z)e^{i\pi z}}{q(z)} = \frac{ze^{i\pi z}}{1-z^2} = -\frac{ze^{i\pi z}}{(z-1)(z+1)}$$

are simple poles at 1 and -1, which lie on the real axis. Using the Cover-up Rule, we obtain

$$\operatorname{Res}(f,1) = -\frac{e^{i\pi}}{2} = \frac{1}{2},$$

$$\operatorname{Res}(f,-1) = -\frac{-e^{-i\pi}}{-2} = \frac{1}{2}.$$

Hence, by Theorem 3.4,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{te^{i\pi t}}{1-t^2}\,dt &= 2\pi i S + \pi i T\\ &= 2\pi i \times 0 + \pi i\left(\tfrac{1}{2} + \tfrac{1}{2}\right) = \pi i.\end{aligned}$$

On equating imaginary parts we obtain

$$\int_{-\infty}^{\infty} \frac{t\sin \pi t}{1-t^2}\,dt = \pi.$$

Solution to Exercise 4.1

The function $h(z) = 1/z^4$ is an even function and it is analytic on $\mathbb{C}$ apart from a pole at 0.

The residue of the function

$$f(z) = (\pi \cot \pi z)/z^4$$

at 0 can be obtained from the Laurent series about 0 for f. Using equation (4.2), we see that

$$\begin{aligned}\frac{\pi\cot\pi z}{z^4} &= \frac{\pi}{z^4}\left(\frac{1}{\pi z} - \frac{1}{3}(\pi z) - \frac{1}{45}(\pi z)^3 - \cdots\right)\\ &= \frac{1}{z^5} - \frac{\pi^2}{3z^3} - \frac{\pi^4}{45z} - \cdots,\end{aligned}$$

so $\operatorname{Res}(f,0) = -\pi^4/45$.

We now check condition (4.1). If z lies on S_N, then $|z| \geq N + \frac{1}{2}$, so, by Lemma 4.1,

$$|f(z)| = \left|\frac{\pi \cot \pi z}{z^4}\right| \leq \frac{2\pi}{\left(N+\frac{1}{2}\right)^4}, \quad \text{for } z \in S_N.$$

Hence, by the Estimation Theorem,

$$\begin{aligned}\left|\int_{S_N} f(z)\,dz\right| &\leq \frac{2\pi}{\left(N+\frac{1}{2}\right)^4} \times 4(2N+1)\\ &= \frac{128\pi}{(2N+1)^3},\end{aligned}$$

which tends to 0 as $N \to \infty$. Thus condition (4.1) holds. It follows from Theorem 4.1 that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = -\frac{1}{2}\operatorname{Res}(f,0) = \frac{\pi^4}{90}.$$

Solution to Exercise 4.2

The function $h(z) = 1/(4z^2-1)$ is an even function and it is analytic on $\mathbb{C}$ apart from simple poles at $\frac{1}{2}$ and $-\frac{1}{2}$.

The residues of the function

$$f(z) = (\pi \cot \pi z)/(4z^2-1)$$

at $\frac{1}{2}$ and $-\frac{1}{2}$ are both 0 (see Exercise 1.5(b)). Since h is analytic at 0,

$$\operatorname{Res}(f,0) = h(0) = -1.$$

We now check condition (4.1). If z lies on S_N, then $|z| \geq N + \frac{1}{2}$, so, by Lemma 4.1 and the backwards form of the Triangle Inequality,

$$|f(z)| = \left|\frac{\pi \cot \pi z}{4z^2-1}\right| \leq \frac{2\pi}{4\left(N+\frac{1}{2}\right)^2-1},$$

for $z \in S_N$. Hence, by the Estimation Theorem,

$$\begin{aligned}\left|\int_{S_N} f(z)\,dz\right| &\leq \frac{2\pi}{4\left(N+\frac{1}{2}\right)^2-1} \times 4(2N+1)\\ &= \frac{8\pi(2N+1)}{(2N+1)^2-1},\end{aligned}$$

which tends to 0 as $N \to \infty$. Thus condition (4.1) holds.

It then follows from Theorem 4.1 that

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\\ &= -\tfrac{1}{2}\left(\operatorname{Res}(f,0) + \operatorname{Res}\left(f,\tfrac{1}{2}\right) + \operatorname{Res}\left(f,-\tfrac{1}{2}\right)\right)\\ &= -\tfrac{1}{2}(-1+0+0) = \tfrac{1}{2}.\end{aligned}$$

Alternatively, the sum can be found by more elementary means. Observe that

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{4n^2-1} &= \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\\ &= \frac{1}{2}\left(\sum_{n=1}^{N}\frac{1}{2n-1} - \sum_{n=1}^{N}\frac{1}{2n+1}\right)\\ &= \frac{1}{2}\left(\sum_{n=1}^{N}\frac{1}{2n-1} - \sum_{m=2}^{N+1}\frac{1}{2m-1}\right)\\ &= \frac{1}{2}\left(1 - \frac{1}{2N+1}\right).\end{aligned}$$

Since $1/(2N+1) \to 0$ as $N \to \infty$, we see that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.$$

Solution to Exercise 4.3

The function $h(z) = 1/z^2$ is an even function and it is analytic on $\mathbb{C}$ apart from a pole at 0.

The residue of the function

$$f(z) = (\pi \operatorname{cosec} \pi z)/z^2$$

at 0 can be obtained from the Laurent series about 0 for f. We see that

$$\begin{aligned}\frac{\pi \operatorname{cosec} \pi z}{z^2} &= \frac{\pi}{z^2}\left(\frac{1}{\pi z} + \frac{1}{6}\pi z + \cdots\right)\\ &= \frac{1}{z^3} + \frac{\pi^2}{6z} + \cdots,\end{aligned}$$

so $\operatorname{Res}(f,0) = \pi^2/6$.

We now check condition (4.3). If z lies on S_N, then $|z| \geq N + \frac{1}{2}$, so, by Lemma 4.2,

$$|f(z)| = \left|\frac{\pi \operatorname{cosec} \pi z}{z^2}\right| \leq \frac{\pi}{\left(N+\frac{1}{2}\right)^2},$$

for $z \in S_N$. Hence, by the Estimation Theorem,

$$\begin{aligned}\left|\int_{S_N} f(z)\,dz\right| &\leq \frac{\pi}{\left(N+\frac{1}{2}\right)^2} \times 4(2N+1) \\ &= \frac{16\pi}{2N+1},\end{aligned}$$

which tends to 0 as $N \to \infty$. Thus condition (4.3) holds.

It follows from Theorem 4.2 that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{1}{2}\operatorname{Res}(f,0) = -\frac{\pi^2}{12}.$$

Alternatively, the sum can be determined from the formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

found in Example 4.1. To see this, observe that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} &= 2\sum_{\substack{n=1\\ n \text{ even}}}^{\infty} \frac{1}{n^2} \\ &= 2\sum_{m=1}^{\infty} \frac{1}{(2m)^2} \\ &= \frac{1}{2}\sum_{m=1}^{\infty} \frac{1}{m^2} \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}.\end{aligned}$$

Rearranging this equation, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2} \times \frac{\pi^2}{6} = -\frac{\pi^2}{12}.$$

Solution to Exercise 4.4

(a) The function $h(z) = 1/(4z^2+1)$ is an even function and it is analytic on $\mathbb{C}$ apart from simple poles at $\frac{1}{2}i$ and $-\frac{1}{2}i$.

The residues of the function

$$f(z) = (\pi \cot \pi z)/(4z^2+1)$$

at $\frac{1}{2}i$ and $-\frac{1}{2}i$ were found in Exercise 1.5(d) to be

$$\operatorname{Res}\left(f, \tfrac{1}{2}i\right) = \operatorname{Res}\left(f, -\tfrac{1}{2}i\right) = -\frac{\pi}{4}\coth \pi/2.$$

Since h is analytic at 0,

$$\operatorname{Res}(f,0) = h(0) = 1.$$

We now check condition (4.1). If z lies on S_N, then $|z| \geq N + \frac{1}{2}$, so, by Lemma 4.1 and the backwards form of the Triangle Inequality,

$$|f(z)| = \left|\frac{\pi \cot \pi z}{4z^2+1}\right| \leq \frac{2\pi}{4\left(N+\frac{1}{2}\right)^2 - 1},$$

for $z \in S_N$. Hence, by the Estimation Theorem,

$$\begin{aligned}\left|\int_{S_N} f(z)\,dz\right| &\leq \frac{2\pi}{4\left(N+\frac{1}{2}\right)^2 - 1} \times 4(2N+1) \\ &= \frac{8\pi(2N+1)}{(2N+1)^2 - 1},\end{aligned}$$

which tends to 0 as $N \to \infty$. Thus condition (4.1) holds.

It follows from Theorem 4.1 that

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{1}{4n^2+1} \\ &= -\frac{1}{2}\left(\operatorname{Res}(f,0) + \operatorname{Res}\left(f,\tfrac{1}{2}i\right) + \operatorname{Res}\left(f,-\tfrac{1}{2}i\right)\right) \\ &= -\frac{1}{2}\left(1 - \frac{\pi}{4}\coth \pi/2 - \frac{\pi}{4}\coth \pi/2\right) \\ &= \frac{\pi}{4}\coth \pi/2 - \frac{1}{2}.\end{aligned}$$

(b) The function $h(z) = 1/(4z^2+1)$ is an even function and it is analytic on $\mathbb{C}$ apart from simple poles at $\frac{1}{2}i$ and $-\frac{1}{2}i$.

The residues of the function

$$f(z) = (\pi \operatorname{cosec} \pi z)/(4z^2+1)$$

at $\frac{1}{2}i$ and $-\frac{1}{2}i$ were found in Exercise 1.5(c) to be

$$\operatorname{Res}\left(f, \tfrac{1}{2}i\right) = \operatorname{Res}\left(f, -\tfrac{1}{2}i\right) = -\frac{\pi}{4\sinh \pi/2}.$$

Since h is analytic at 0,

$$\operatorname{Res}(f,0) = h(0) = 1.$$

We now check condition (4.3). If z lies on S_N, then $|z| \geq N + \frac{1}{2}$, so, by Lemma 4.2 and the backwards form of the Triangle Inequality,

$$|f(z)| = \left|\frac{\pi \operatorname{cosec} \pi z}{4z^2+1}\right| \leq \frac{\pi}{4\left(N+\frac{1}{2}\right)^2 - 1},$$

for $z \in S_N$. Hence, by the Estimation Theorem,

$$\begin{aligned}\left|\int_{S_N} f(z)\,dz\right| &\leq \frac{\pi}{4\left(N+\frac{1}{2}\right)^2 - 1} \times 4(2N+1) \\ &= \frac{4\pi(2N+1)}{(2N+1)^2-1},\end{aligned}$$

which tends to 0 as $N \to \infty$. Thus condition (4.3) holds.

It follows from Theorem 4.2 that

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2+1} \\ &= -\frac{1}{2}\left(\operatorname{Res}(f,0) + \operatorname{Res}\left(f, \tfrac{1}{2}i\right) + \operatorname{Res}\left(f, -\tfrac{1}{2}i\right)\right) \\ &= -\frac{1}{2}\left(1 - \frac{\pi}{4\sinh \pi/2} - \frac{\pi}{4\sinh \pi/2}\right) \\ &= \frac{\pi}{4\sinh \pi/2} - \frac{1}{2}.\end{aligned}$$

Solution to Exercise 4.5

Let $\alpha \in \mathbb{C} - \mathbb{Z}$. Then the function

$$h(z) = \frac{1}{z^2 - \alpha^2}$$

is an even function and it is analytic on $\mathbb{C}$ apart from simple poles at α and $-\alpha$, which are not integers.

We now calculate the residues of the function

$$f(z) = \frac{\pi \cot \pi z}{z^2-\alpha^2} = \frac{\pi \cot \pi z}{(z-\alpha)(z+\alpha)}$$

at 0, α and $-\alpha$. Since h is analytic at 0,

$$\operatorname{Res}(f,0) = h(0) = -1/\alpha^2.$$

Also, using the Cover-up Rule, we obtain

$$\begin{aligned}\operatorname{Res}(f,\alpha) &= \frac{\pi \cot \pi\alpha}{2\alpha}, \\ \operatorname{Res}(f,-\alpha) &= \frac{\pi \cot(-\pi\alpha)}{-2\alpha} = \frac{\pi \cot \pi\alpha}{2\alpha}.\end{aligned}$$

Next we check condition (4.1). If z lies on S_N, then $|z| \geq N + \frac{1}{2}$, so, by Lemma 4.1 and the backwards form of the Triangle Inequality,

$$\begin{aligned}|f(z)| &= \left|\frac{\pi \cot \pi z}{z^2-\alpha^2}\right| \\ &\leq \frac{2\pi}{|z|^2 - |\alpha|^2} \quad (\text{for } |z| > |\alpha|) \\ &\leq \frac{2\pi}{\left(N+\frac{1}{2}\right)^2 - |\alpha|^2},\end{aligned}$$

for $z \in S_N$, $N + \frac{1}{2} > |\alpha|$. Hence, by the Estimation Theorem,

$$\begin{aligned}\left|\int_{S_N} f(z)\,dz\right| &\leq \frac{2\pi}{\left(N+\frac{1}{2}\right)^2 - |\alpha|^2} \times 4(2N+1) \\ &= \frac{32\pi(2N+1)}{(2N+1)^2 - 4|\alpha|^2},\end{aligned}$$

for $N + \frac{1}{2} > |\alpha|$, which tends to 0 as $N \to \infty$. Thus condition (4.1) holds.

It follows from Theorem 4.1 that

$$\begin{aligned}\sum_{n=1}^{\infty} h(n) &= \sum_{n=1}^{\infty} \frac{1}{n^2-\alpha^2} \\ &= -\frac{1}{2}(\operatorname{Res}(f,0) + \operatorname{Res}(f,\alpha) + \operatorname{Res}(f,-\alpha)) \\ &= -\frac{1}{2}\left(-\frac{1}{\alpha^2} + \frac{\pi \cot \pi\alpha}{2\alpha} + \frac{\pi \cot \pi\alpha}{2\alpha}\right) \\ &= \frac{1}{2\alpha^2} - \frac{\pi \cot \pi\alpha}{2\alpha}.\end{aligned}$$

Rearranging this, we obtain

$$\begin{aligned}\pi \cot \pi\alpha &= \frac{1}{\alpha} - 2\alpha \sum_{n=1}^{\infty} \frac{1}{n^2-\alpha^2} \\ &= \frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2-n^2},\end{aligned}$$

as required.

Remark: Notice that this equation can be written in the form

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}, \quad \text{for } z \in \mathbb{C} - \mathbb{Z},$$

which gives a representation of the function $z \longmapsto \pi \cot \pi z$ as the sum of an infinite series of rational functions, valid for all points in the domain of this function.

Solution to Exercise 5.1

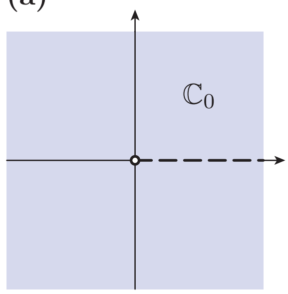
(a) $\mathrm{Arg}_\pi(i) = \pi/2$, since $\pi/2$ is the argument of i that lies in $(-\pi, \pi]$.

(b) $\mathrm{Arg}_0(-1) = -\pi$, since $-\pi$ is the argument of -1 that lies in $(-2\pi, 0]$.

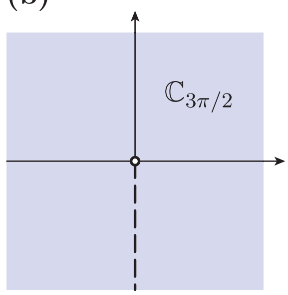
(c) $\mathrm{Arg}_{3\pi/2}(1-i) = -\pi/4$, since $-\pi/4$ is the argument of $1-i$ that lies in $(-\pi/2, 3\pi/2]$.

Solution to Exercise 5.2

(a)



(b)



Solution to Exercise 5.3

(a) We have

$$\begin{aligned}\mathrm{Log}_{3\pi}(-i) &= \log|-i| + i\,\mathrm{Arg}_{3\pi}(-i)\\ &= 0 + \frac{3\pi}{2}i,\end{aligned}$$

since $3\pi/2$ is an argument of $-i$ and $\pi < 3\pi/2 \leq 3\pi$.

(b) We have

$$\begin{aligned}\mathrm{Log}_{2\pi}(2) &= \log 2 + i\,\mathrm{Arg}_{2\pi}(2)\\ &= \log 2 + 2\pi i,\end{aligned}$$

since 2π is an argument of 2 and $0 < 2\pi \leq 2\pi$.

(c) We have

$$\begin{aligned}\mathrm{Log}_{3\pi/2}(-3) &= \log|-3| + i\,\mathrm{Arg}_{3\pi/2}(-3)\\ &= \log 3 + \pi i,\end{aligned}$$

since π is an argument of -3 and $-\pi/2 < \pi \leq 3\pi/2$.

Solution to Exercise 5.4

(a) Since $\displaystyle\sum_{n=0}^{\infty}(2z)^n$ is a geometric series with sum

$$\frac{1}{1-2z}, \quad \text{for } |z| < \tfrac{1}{2},$$

we deduce that the function

$$g(z) = \frac{1}{1-2z} \quad \left(z \in \mathbb{C} - \{\tfrac{1}{2}\}\right)$$

is an analytic extension of f to $\mathbb{C} - \{\frac{1}{2}\}$.

(b) Since

$$\mathrm{Log}(1+w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \cdots,$$

for $|w| < 1$, we see that

$$\begin{aligned}\mathrm{Log}\, z &= \mathrm{Log}(1 + (z-1))\\ &= (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n,\end{aligned}$$

for $|z-1| < 1$. Therefore the function

$$g(z) = \mathrm{Log}\, z \quad (z \in \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\})$$

is an analytic extension of f to $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$.

Solution to Exercise 5.5

(a) Here $\mathcal{R} = \{z : |z| < 1\}$ and $\mathcal{S} = \mathbb{C} - \{1\}$. Using the binomial series for $(1-z)^{-2}$, we have

$$\begin{aligned}(1-z)^{-2} &= 1 + 2z + 3z^2 + \cdots\\ &= \sum_{n=1}^{\infty} nz^{n-1}, \quad \text{for } |z| < 1,\end{aligned}$$

so f and g agree on the region

$$\mathcal{R} = \{z : |z| < 1\} \subseteq \mathcal{S}.$$

Hence g is a direct analytic continuation of f from $\mathcal{R}$ to $\mathcal{S}$.

(b) Here $\mathcal{R} = \mathbb{C}_{2\pi}$ and $\mathcal{S} = \mathbb{C}_{3\pi/2}$. Since f and g agree on the region $\mathcal{T} = \{z : \operatorname{Re} z < 0\} \subseteq \mathcal{R} \cap \mathcal{S}$, we deduce that f and g are direct analytic continuations of each other.

Solution to Exercise 5.6

First note that f is the principal square root function,

$$f(z) = \sqrt{z} = \exp\left(\tfrac{1}{2}\operatorname{Log}_\pi(z)\right) \quad (z \in \mathbb{C}_\pi),$$

since $\operatorname{Log}_\pi = \operatorname{Log}$. Let us then consider the function

$$g(z) = \exp\left(\tfrac{1}{2}\operatorname{Log}_{3\pi/2}(z)\right) \quad (z \in \mathbb{C}_{3\pi/2}),$$

chosen because $z \longmapsto \operatorname{Log}_{3\pi/2}(z)$ is a direct analytic continuation of $z \longmapsto \operatorname{Log}_\pi(z)$ (see Example 5.2(b)). Since

$$f(z) = g(z), \quad \text{for } z \in \mathcal{T},$$

where $\mathcal{T} = \{z : \operatorname{Re} z > 0\} \subseteq \mathbb{C}_\pi \cap \mathbb{C}_{3\pi/2}$, we deduce that g is a direct analytic continuation of f from $\mathbb{C}_\pi$ to $\mathbb{C}_{3\pi/2}$.

Solution to Exercise 5.7

(a) Since

$$\frac{1}{t} \le \frac{1}{\sqrt{t}}, \quad \text{for } t \ge 1,$$

it follows from the Monotonicity Inequality (Theorem 1.3(f) of Unit B1) that if $r > 1$, then

$$\int_1^r \frac{1}{t}\,dt \le \int_1^r \frac{1}{\sqrt{t}}\,dt.$$

Evaluating each side of this inequality, we obtain

$$\left[\log t\right]_1^r \le \left[2\sqrt{t}\right]_1^r,$$

that is,

$$\log r \le 2\sqrt{r} - 2 \le 2\sqrt{r}, \quad \text{for } r > 1. \tag{S4}$$

(b) **(i)** Using inequality (S4), we have

$$0 < \frac{\log r}{r} \le \frac{2}{\sqrt{r}}, \quad \text{for } r > 1.$$

Since $\lim_{r\to\infty} \dfrac{2}{\sqrt{r}} = 0$, it follows that

$$\lim_{r\to\infty} \frac{\log r}{r} = 0.$$

(ii) Let $\varepsilon = 1/r$. Then, from inequality (S4),

$$\log 1/\varepsilon \le 2\sqrt{1/\varepsilon}, \quad \text{for } 0 < \varepsilon < 1.$$

Hence, multiplying through by ε, we obtain

$$-\varepsilon \log \varepsilon \le 2\sqrt{\varepsilon}, \quad \text{for } 0 < \varepsilon < 1,$$

so

$$-2\sqrt{\varepsilon} \le \varepsilon \log \varepsilon < 0, \quad \text{for } 0 < \varepsilon < 1.$$

Since $\lim_{\varepsilon\to 0}(-2\sqrt{\varepsilon}) = 0$, it follows that

$$\lim_{\varepsilon\to 0} \varepsilon \log \varepsilon = 0.$$

Solution to Exercise 5.8

We find the integral using the five-step strategy.

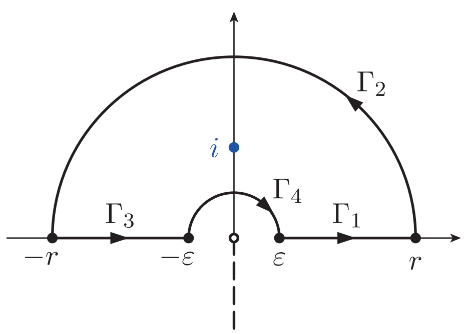
1. Let

$$f(z) = \frac{\exp\left(-\frac{1}{2}\operatorname{Log}_{3\pi/2}(z)\right)}{z^2+1},$$

which satisfies $f(t) = t^{-1/2}/(t^2+1)$, for $t > 0$. The function f is analytic on the simply connected cut plane $\mathbb{C}_{3\pi/2}$, except for a simple pole at i (the point $-i$ lies on the negative imaginary axis, which is excluded from the cut plane). We choose the contour integral

$$I = \int_\Gamma f(z)\,dz,$$

where $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ is the contour shown in the figure (with $0 < \varepsilon < 1$ and $r > 1$).



2. Since

$$f(z) = \frac{\exp\left(-\frac{1}{2}\operatorname{Log}_{3\pi/2}(z)\right)}{(z-i)(z+i)},$$

we can obtain the residue of f at i using the Cover-up Rule. To do this, first observe that

$$\begin{aligned}\operatorname{Log}_{3\pi/2}(i) &= \log|i| + i\operatorname{Arg}_{3\pi/2}(i)\\ &= \log 1 + i\pi/2 = i\pi/2.\end{aligned}$$

Hence

$$\operatorname{Res}(f,i) = \frac{\exp\left(-\frac{1}{2}\operatorname{Log}_{3\pi/2}(i)\right)}{2i} = \frac{e^{-i\pi/4}}{2i}.$$

So, by the Residue Theorem,

$$I = 2\pi i \operatorname{Res}(f,i) = 2\pi i \times \frac{e^{-i\pi/4}}{2i} = \frac{\pi}{e^{i\pi/4}}.$$

3. Splitting up the integral gives

$$\begin{aligned} I &= \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz \\ &\quad + \int_{\Gamma_3} f(z)\,dz + \int_{\Gamma_4} f(z)\,dz. \end{aligned} \tag{S5}$$

The first integral is

$$\int_{\Gamma_1} f(z)\,dz = \int_\varepsilon^r f(t)\,dt = \int_\varepsilon^r \frac{t^{-1/2}}{t^2+1}\,dt.$$

For the third integral, note that

$$\operatorname{Log}_{3\pi/2}(t) = \log|t| + i\pi,$$

for $t < 0$, so

$$\begin{aligned} \exp\left(-\tfrac{1}{2}\operatorname{Log}_{3\pi/2}(t)\right) &= \exp\left(-\tfrac{1}{2}(\log|t| + i\pi)\right) \\ &= \exp\left(-\tfrac{1}{2}\log|t|\right)\exp\left(-\tfrac{1}{2}i\pi\right) \\ &= -i|t|^{-1/2}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\Gamma_3} f(z)\,dz &= -i\int_{-r}^{-\varepsilon} \frac{|t|^{-1/2}}{t^2+1}\,dt \\ &= -i\int_\varepsilon^r \frac{t^{-1/2}}{t^2+1}\,dt, \end{aligned}$$

since $t \longmapsto |t|^{-1/2}/(t^2+1)$ is an even function.

4. In order to obtain upper estimates for the moduli of the remaining integrals, we first observe that

$$\begin{aligned} &\left|\exp\left(-\tfrac{1}{2}\operatorname{Log}_{3\pi/2}(z)\right)\right| \\ &= \left|\exp\left(-\tfrac{1}{2}(\log|z| + i\operatorname{Arg}_{3\pi/2}(z))\right)\right| \\ &= \left|\exp\left(-\tfrac{1}{2}\log|z|\right)\right| = |z|^{-1/2}. \end{aligned}$$

Also, by the backwards form of the Triangle Inequality, we have $|z^2+1| \geq |z^2| - 1 = r^2 - 1$, for $z \in \Gamma_2$, and similarly $|z^2+1| \geq 1 - \varepsilon^2$, for $z \in \Gamma_4$. Hence

$$|f(z)| \leq \frac{r^{-1/2}}{r^2-1}, \quad \text{for } z \in \Gamma_2,$$

$$|f(z)| \leq \frac{\varepsilon^{-1/2}}{1-\varepsilon^2}, \quad \text{for } z \in \Gamma_4.$$

Since f is continuous on Γ_2 and Γ_4, we can apply the Estimation Theorem to give

$$\left|\int_{\Gamma_2} f(z)\,dz\right| \leq \frac{r^{-1/2}}{r^2-1} \times \pi r = \frac{\pi r^{1/2}}{r^2-1},$$

$$\left|\int_{\Gamma_4} f(z)\,dz\right| \leq \frac{\varepsilon^{-1/2}}{1-\varepsilon^2} \times \pi\varepsilon = \frac{\pi\varepsilon^{1/2}}{1-\varepsilon^2},$$

for $0 < \varepsilon < 1$ and $r > 1$.

5. Using these estimates, we see that the Γ_2 integral tends to 0 as $r \to \infty$, and the Γ_4 integral tends to 0 as $\varepsilon \to 0$. Taking the limits as $r \to \infty$ and as $\varepsilon \to 0$ of each expression in equation (S5), and using the values for I and the integrals that we have found, we obtain

$$\int_0^\infty \frac{t^{-1/2}}{t^2+1}\,dt - i\int_0^\infty \frac{t^{-1/2}}{t^2+1}\,dt = \frac{\pi}{e^{i\pi/4}}.$$

Therefore

$$\begin{aligned} \int_0^\infty \frac{t^{-1/2}}{t^2+1}\,dt &= \frac{\pi}{(1-i)e^{i\pi/4}} \\ &= \frac{\pi}{(1-i)\times\frac{1}{\sqrt{2}}(1+i)} = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

Solution to Exercise 5.9

The improper integral

$$\int_0^\infty \frac{t^a}{t^2-t}\,dt, \quad \text{where } 0 < a < 1,$$

satisfies the hypotheses of Theorem 5.3 with $p(z) = 1$ and $q(z) = z^2 - z$, because the degree of q exceeds that of p by 2, and the poles of p/q on the non-negative real axis are simple ones, at 0 and 1.

Since q has no zeros in $\mathbb{C}_{2\pi} = \mathbb{C} - \{x \in \mathbb{R} : x \geq 0\}$, we have $S = 0$. The only zero of q on the positive real axis is a simple zero at 1, which gives rise to a simple pole of

$$f_2(z) = \frac{\exp(a\operatorname{Log} z)}{z^2 - z} = \frac{\exp(a\operatorname{Log} z)}{z(z-1)}$$

at 1. Hence

$$T = \operatorname{Res}(f_2, 1) = \frac{\exp(a\operatorname{Log} 1)}{1} = 1,$$

by the Cover-up Rule. Thus, by Theorem 5.3,

$$\int_0^\infty \frac{t^a}{t^2-t}\,dt = -\pi\cot\pi a.$$

Solution to Exercise 5.10

(a) Since

$$\begin{aligned}\operatorname{Log}_\pi(z) &= \operatorname{Log}_{2\pi}(z), \quad \text{for } \operatorname{Im} z > 0,\\ \operatorname{Log}_{2\pi}(z) &= \operatorname{Log}_{3\pi}(z), \quad \text{for } \operatorname{Im} z < 0,\end{aligned}$$

we deduce that

$$\begin{aligned}f_1(z) &= f_2(z), \quad \text{for } \operatorname{Im} z > 0,\\ f_2(z) &= f_3(z), \quad \text{for } \operatorname{Im} z < 0.\end{aligned}$$

Since the region $\{z : \operatorname{Im} z > 0\} \subseteq \mathbb{C}_\pi \cap \mathbb{C}_{2\pi}$ and the region $\{z : \operatorname{Im} z < 0\} \subseteq \mathbb{C}_{2\pi} \cap \mathbb{C}_{3\pi}$, it follows that $(f_{k+1}, \mathbb{C}_{(k+1)\pi})$ is a direct analytic continuation of $(f_k, \mathbb{C}_{k\pi})$, for $k = 1, 2$. Also, $\mathbb{C}_\pi = \mathbb{C}_{3\pi}$.

Thus $(f_1, \mathbb{C}_\pi), (f_2, \mathbb{C}_{2\pi}), (f_3, \mathbb{C}_{3\pi})$ is a closed chain of functions.

Now, for $z \in \mathbb{C}_\pi = \mathbb{C}_{3\pi}$,

$$\begin{aligned}f_3(z) &= \exp\left(\tfrac{1}{2} \operatorname{Log}_{3\pi}(z)\right)\\ &= \exp\left(\tfrac{1}{2}(\operatorname{Log}_\pi(z) + 2\pi i)\right)\\ &= e^{\pi i} \exp\left(\tfrac{1}{2}(\operatorname{Log}_\pi(z)\right)\\ &= -f_1(z).\end{aligned}$$

Thus $f_1 \neq f_3$.

(b) Since

$$\begin{aligned}\operatorname{Log}_\pi(z) &= \operatorname{Log}_{2\pi}(z), \quad \text{for } \operatorname{Im} z > 0,\\ \operatorname{Log}_{2\pi}(z) &= \operatorname{Log}_{3\pi}(z), \quad \text{for } \operatorname{Im} z < 0,\\ \operatorname{Log}_{3\pi}(z) &= \operatorname{Log}_{4\pi}(z), \quad \text{for } \operatorname{Im} z > 0,\\ \operatorname{Log}_{4\pi}(z) &= \operatorname{Log}_{5\pi}(z), \quad \text{for } \operatorname{Im} z < 0,\end{aligned}$$

we deduce that

$$\begin{aligned}f_1(z) &= f_2(z), \quad \text{for } \operatorname{Im} z > 0,\\ f_2(z) &= f_3(z), \quad \text{for } \operatorname{Im} z < 0,\\ f_3(z) &= f_4(z), \quad \text{for } \operatorname{Im} z > 0,\\ f_4(z) &= f_5(z), \quad \text{for } \operatorname{Im} z < 0.\end{aligned}$$

Hence $(f_{k+1}, \mathbb{C}_{(k+1)\pi})$ is a direct analytic continuation of $(f_k, \mathbb{C}_{k\pi})$, for $k = 1, 2, 3, 4$. Also, $\mathbb{C}_\pi = \mathbb{C}_{5\pi}$. Thus

$$(f_1, \mathbb{C}_\pi),\ (f_2, \mathbb{C}_{2\pi}),\ (f_3, \mathbb{C}_{3\pi}),\ (f_4, \mathbb{C}_{4\pi}),\ (f_5, \mathbb{C}_{5\pi})$$

is a closed chain of functions.

Now, for $z \in \mathbb{C}_\pi = \mathbb{C}_{5\pi}$,

$$\begin{aligned}f_5(z) &= \exp\left(\tfrac{1}{2} \operatorname{Log}_{5\pi}(z)\right)\\ &= \exp\left(\tfrac{1}{2}(\operatorname{Log}_\pi(z) + 4\pi i)\right)\\ &= e^{2\pi i} \exp\left(\tfrac{1}{2} \operatorname{Log}_\pi(z)\right)\\ &= f_1(z).\end{aligned}$$

Thus $f_1 = f_5$.

Solution to Exercise 5.11

Let

$$h(z) = \frac{1}{1-z} \quad (z \in \mathbb{C} - \{1\}).$$

Then, from Example 5.1, $(h, \mathbb{C} - \{1\})$ is a direct analytic continuation of (g, D_2). Also, since

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad \text{for } |z| < 1,$$

we see that $(h, \mathbb{C} - \{1\})$ is a direct analytic continuation of (f, D_1).

Thus $(f, D_1), (h, \mathbb{C} - \{1\}), (g, D_2)$ is a chain of functions. Since $D_1 \cap D_2 = \varnothing$, we see that f and g are not direct analytic continuations of each other, so they must be indirect analytic continuations of each other.

Solution to Exercise 5.12

The power series defining f is a geometric series with common ratio $z + 1$, so

$$f(z) = \frac{z+1}{1-(z+1)} = -1 - \frac{1}{z}, \quad \text{for } |z+1| < 1.$$

Since f and g agree on the region $\{z : |z+1| < 1\} \subseteq \mathbb{C} - \{0\}$, we deduce that g is a direct analytic continuation of f from $\{z : |z+1| < 1\}$ to $\mathbb{C} - \{0\}$.

Solution to Exercise 5.13

(a) Since the given power series has disc of convergence $\{z : |z| < 1\}$, we deduce, by the Differentiation Rule for power series (Theorem 2.3 of Unit B3), that

$$f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n}, \quad \text{for } |z| < 1, \tag{S6}$$

and hence that

$$zf'(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\operatorname{Log}(1-z), \quad \text{for } |z| < 1.$$

Thus

$$f'(z) = \frac{-\operatorname{Log}(1-z)}{z}, \quad \text{for } 0 < |z| < 1,$$

as required.

(b) The analytic function

$$g(z) = \frac{-\operatorname{Log}(1-z)}{z}$$
$$(z \in \mathbb{C} - \{x \in \mathbb{R} : x = 0 \text{ or } x \geq 1\})$$

has a removable singularity at 0, which can be removed by putting $g(0) = f'(0) = 1$ (from equation (S6)). Then g and f' agree on $\{z : |z| < 1\}$, so g is a direct analytic continuation of f' from $\{z : |z| < 1\}$ to $\mathbb{C} - \{x \in \mathbb{R} : x \geq 1\}$.

(c) Since $\mathbb{C} - \{x \in \mathbb{R} : x \geq 1\}$ is simply connected, g has a primitive G on this region, by the Primitive Theorem, and, by adding a constant if necessary, we can arrange that $G(0) = f(0) = 0$. Then both G and f are primitives of f' on $\{z : |z| < 1\}$ that agree at 0. Hence

$$G(z) = f(z), \quad \text{for } |z| < 1,$$

so G is a direct analytic continuation of f from $\{z : |z| < 1\}$ to $\mathbb{C} - \{x \in \mathbb{R} : x \geq 1\}$, which is a larger region.

Solution to Exercise 5.14

Observe that

$$\int_0^\infty \frac{t^{3/2}}{(t^2+1)(t-1)}\,dt = \int_0^\infty \frac{t}{(t^2+1)(t-1)} t^{1/2}\,dt.$$

Therefore we can apply Theorem 5.3 with $a = \frac{1}{2}$, $p(z) = z$ and $q(z) = \left(z^2+1\right)(z-1)$. To see that the conditions of the theorem are satisfied, observe that the degree of q exceeds that of p by 2, and the poles 1, i and $-i$ of p/q are all simple.

Of these poles, only i and $-i$ lie in $\mathbb{C}_{2\pi}$, and only 1 lies on the positive real axis. Let

$$\begin{aligned}
g_1(z) &= z\exp\left(\tfrac{1}{2}\operatorname{Log}_{2\pi}(z)\right),\\
g_2(z) &= z\exp\left(\tfrac{1}{2}\operatorname{Log} z\right),\\
h(z) &= q(z) = (z-i)(z+i)(z-1),
\end{aligned}$$

and $f_1(z) = g_1(z)/h(z)$ and $f_2(z) = g_2(z)/h(z)$. Observe that g_1 is analytic at the points i and $-i$, and g_2 is analytic at the point 1. By the Cover-up Rule, we have

$$\begin{aligned}
\operatorname{Res}(f_1, i) &= \frac{i \times \exp\left(\frac{1}{2}\operatorname{Log}_{2\pi}(i)\right)}{2i(i-1)}\\
&= \frac{ie^{i\pi/4}}{-2-2i} = -\frac{i}{2\sqrt{2}},\\
\operatorname{Res}(f_1, -i) &= \frac{-i \times \exp\left(\frac{1}{2}\operatorname{Log}_{2\pi}(-i)\right)}{-2i(-i-1)}\\
&= \frac{-ie^{i3\pi/4}}{-2+2i} = -\frac{i}{2\sqrt{2}},\\
\operatorname{Res}(f_2, 1) &= \frac{1 \times \exp\left(\frac{1}{2}\operatorname{Log} 1\right)}{2} = \frac{1}{2}.
\end{aligned}$$

Hence, by Theorem 5.3,

$$\begin{aligned}
\int_0^\infty \frac{t^{3/2}}{(t^2+1)(t-1)}\,dt &= -\left(\pi e^{-\pi i/2}\operatorname{cosec}\pi/2\right)S\\
&\quad - (\pi\cot\pi/2)T\\
&= \pi i\left(-\frac{i}{2\sqrt{2}} - \frac{i}{2\sqrt{2}}\right)\\
&\quad - 0 \times \frac{1}{2}\\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

Remark: Since $a = \frac{1}{2}$, it follows that $\pi\cot\pi a = 0$, so we did not need to evaluate $T = \operatorname{Res}(f_2, 1)$.

Solution to Exercise 5.15

Observe that

$$\begin{aligned}
f(z) &= \frac{1}{1-z}, \quad \text{for } |z| < 1,\\
g(z) &= -\frac{1/z}{1-1/z} = \frac{1}{1-z}, \quad \text{for } |z| > 1.
\end{aligned}$$

Consider the analytic function

$$h(z) = \frac{1}{1-z} \quad (z \in \mathbb{C} - \{1\}).$$

Then f and h agree on the region

$$D_1 = \{z : |z| < 1\} \subseteq \mathbb{C} - \{1\},$$

so h is a direct analytic continuation of f from D_1 to $\mathbb{C} - \{1\}$. Also, g and h agree on the region

$$D_2 = \{z : |z| > 1\} \subseteq \mathbb{C} - \{1\},$$

so h is a direct analytic continuation of g from D_2 to $\mathbb{C} - \{1\}$.

It follows that

$$(f, D_1), (h, \mathbb{C} - \{1\}), (g, D_2)$$

is a chain of functions. The functions f and g are not direct analytic continuations of each other, because $D_1 \cap D_2 = \varnothing$, so they must be indirect analytic continuations of each other.

Unit C2

Zeros and extrema

Introduction

In this unit we investigate two practical questions about analytic functions.

1. What can we say about the number and location of the zeros of a given analytic function?

   For example, how many zeros does the function

   $$f(z) = z^5 + z^3 + iz + 1$$

   have in the open disc $\{z : |z| < 2\}$?

2. How do we determine the maximum (or minimum) value taken by the modulus of a given analytic function on a compact set in the domain of the function?

   For example, what is the maximum value of the function

   $$|f(z)| = |z^2 - z - 1|$$

   on the set $\{z : |z| \leq 1\}$?

In the process of developing methods for answering these questions, we will encounter more theoretical questions such as:

> Is the image of a region under an analytic function also a region?

In Section 1 we introduce a geometric notion called the *winding number* of a closed path Γ around a point α. This counts the number of times that the path Γ winds around the point α. We then show that if Γ is a closed contour, then the winding number of Γ around α can be expressed as a certain contour integral around Γ.

In Section 2 we use the Residue Theorem to prove a key result, called the Argument Principle, which relates the number of zeros of an analytic function f inside a simple-closed contour Γ to the winding number of $f(\Gamma)$ around 0. This leads to various methods for finding the number of zeros of f inside Γ.

Section 3 is devoted to some theoretical consequences of the Argument Principle, which concern the *local* behaviour of analytic functions. Here the word 'local' is used in the same sense as it was used in Unit A3: to refer to a property of a function that depends only on the values taken by the function near to points in its domain. One of the key results in this section is the Open Mapping Theorem, which states that non-constant analytic functions map open sets to open sets.

In Section 4 we use the Open Mapping Theorem to obtain a result called the Maximum Principle. This result simplifies the search for the maximum value of the modulus of an analytic function on a compact set, because it tells us that this maximum value must be attained somewhere on the boundary of the set.

Section 5 introduces the idea of *uniform convergence* of sequences and series of functions. We use this idea to define the zeta function

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots,$$

and prove that it is analytic on the open half-plane $\{z : \operatorname{Re} z > 1\}$.

Finally, in Section 6, we discuss some of the special functions of complex analysis, and investigate their relationship to other parts of mathematics. In particular, we review Riemann's remarkable scheme for applying complex analysis to number theory. This scheme led to a proof of the Prime Number Theorem on the distribution of primes, and bequeathed to future generations an intriguing problem about the zeta function, which remains unresolved to this day.

Unit guide

This unit contains a number of important topics, many of which occur in other mathematical subjects such as topology and real analysis.

Section 1 describes the winding number, which plays a key role in Rouché's Theorem in Section 2.

Sections 3 and 4 are about some of the central results of complex analysis. These results will be applied in later units.

Section 5 is on the relatively advanced topic of uniform convergence. Understanding this material will give you a powerful method for working with and constructing analytic functions such as the zeta function, which is discussed further in Section 6.

1 Winding numbers

After working through this section, you should be able to:

- define the *winding number of a path* around a point and determine it in simple cases
- express the *winding number of a closed contour* around a point as a contour integral.

1.1 Defining winding numbers

Imagine that a dog is tied by its lead to a post, but is otherwise free to roam. The path of the dog during a given time is shown in Figure 1.1. The dog's initial point is the same as its final point, so the path is closed. It seems that the dog has wandered around paying particular attention to a certain stone and also trying (unsuccessfully) to reach a tree.

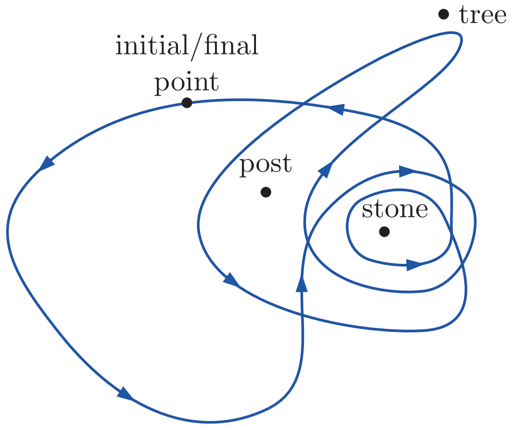


Figure 1.1 The closed path traversed by the dog

In the course of its journey, the dog went twice around the post, in an anticlockwise direction. You can either 'see' this directly or else you can calculate it more systematically, as follows. Consider a ray starting from the post, pointing in any direction. Moving outwards along the ray, each time you cross the path of the dog, count $+1$ if the dog passes from your right to your left (Figure 1.2(a)), and count -1 if the dog passes from your left to your right (Figure 1.2(b)).

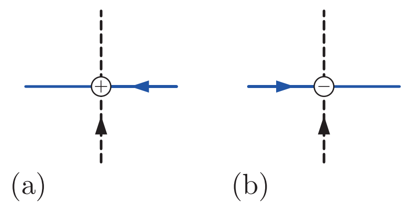


Figure 1.2 Moving along the dashed line, count (a) $+1$ or (b) -1 as the solid line crosses from right to left, or left to right, respectively

In Figure 1.3 the ray from the post points downwards. There are two $+1$ crossings, so the dog *winds* around the post twice anticlockwise.

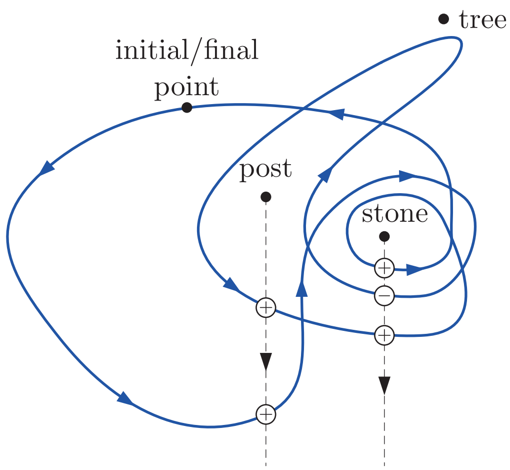


Figure 1.3 Counting crossings along rays

A similar method can be applied to count how many times the dog winds around other objects.

Exercise 1.1

Determine how many times the dog winds around the following objects.

(a) The stone (b) The tree

We frequently encounter paths Γ that are specified by means of their parametrisations γ. But it is not easy to determine from γ how many times the path winds around a given point. So we wish to develop a method for investigating winding systematically. To this end, we first consider a simple example of a path Γ that winds around the point 0.

Consider the closed path

$$\Gamma : \gamma(t) = (2 + \sin t)e^{2it} \quad (t \in [0, 2\pi]),$$

some values of which are plotted in the following table.

t	$\gamma(t)$
0	2
$\pi/4$	$(2 + 1/\sqrt{2})i$
$\pi/2$	-3
$3\pi/4$	$-(2 + 1/\sqrt{2})i$
π	2
$5\pi/4$	$(2 - 1/\sqrt{2})i$
$3\pi/2$	-1
$7\pi/4$	$-(2 - 1/\sqrt{2})i$
2π	2

The path Γ is illustrated in Figure 1.4.

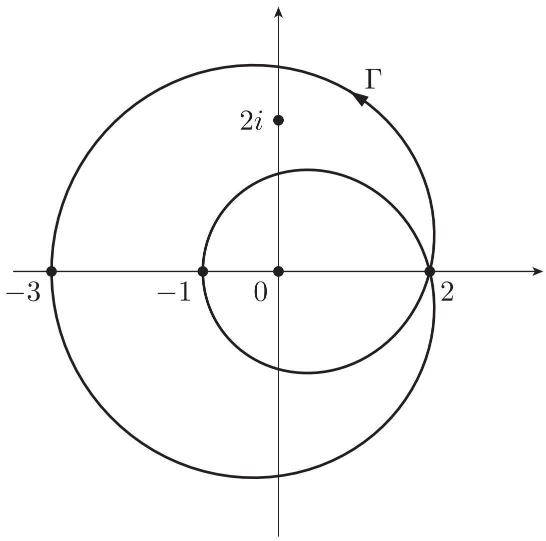


Figure 1.4 The closed path $\Gamma : \gamma(t) = (2 + \sin t)e^{2it}$ $(t \in [0, 2\pi])$

As the parameter t increases from 0 to 2π, the image point $\gamma(t)$ traverses Γ, starting and finishing at the point $\gamma(0) = \gamma(2\pi) = 2$, and winding twice around 0 (anticlockwise) in the process.

The manner in which a path such as Γ winds around 0 depends on the 'angular change' of the ray from 0 through $\gamma(t)$ that occurs as $\gamma(t)$ traverses the path Γ. To measure this 'angular change' for a given path

$$\Gamma : \gamma(t) \quad (t \in [a, b]),$$

we need to choose an argument $\theta(t)$ for each point $\gamma(t)$ on Γ in such a way that the argument $\theta(t)$ varies continuously as t increases from a to b. The 'angular change' along Γ is then equal to the change in the 'argument function'

$$\theta\colon t \longmapsto \theta(t) \quad (t \in [a, b]),$$

given by $\theta(b) - \theta(a)$ (see Figure 1.5). Notice that we cannot simply choose $\theta(t) = \operatorname{Arg}(\gamma(t))$ in general, since this function would jump by 2π as $\gamma(t)$ crossed the negative real axis, so it would fail to be continuous. However, if Γ does not meet this axis, then the choice $\theta(t) = \operatorname{Arg}(\gamma(t))$ is possible.

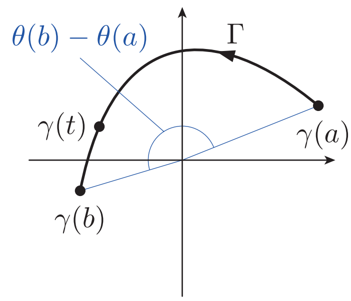


Figure 1.5 Angular change of a path that crosses the negative real axis

Definition

Let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a path lying in $\mathbb{C} - \{0\}$.

A **continuous argument function** for Γ is a continuous function

$$\theta\colon [a, b] \longrightarrow \mathbb{R}$$

such that, for each $t \in [a, b]$, $\theta(t)$ is an argument of $\gamma(t)$.

Note that Γ need not be a closed path in this definition.

Since $\theta(t)$ is an argument of $\gamma(t)$, we can write

$$\gamma(t) = |\gamma(t)|e^{i\theta(t)}, \quad \text{for } t \in [a, b],$$

so

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{i\theta(t)}, \quad \text{for } t \in [a, b].$$

Reversing this reasoning, if for a given path $\Gamma : \gamma(t)$ $(t \in [a, b])$ we can express $\gamma(t)/|\gamma(t)|$ in the form $e^{i\theta(t)}$, where θ is a continuous function on $[a, b]$, then θ must be a continuous argument function for Γ.

Example 1.1

Determine a continuous argument function for the path

$$\Gamma : \gamma(t) = 2e^{it} \quad (t \in [0, \pi]).$$

Solution

Since $|\gamma(t)| = 2$, for $t \in [0, \pi]$, we have

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{it}, \quad \text{for } t \in [0, \pi].$$

Thus one choice of continuous argument function is

$$\theta(t) = t \quad (t \in [0, \pi]).$$

Remark

The solution to Example 1.1 is not unique. Indeed, we could have chosen $\theta(t) = t + 2\pi$ or, in general,

$$\theta(t) = t + 2n\pi \quad (t \in [0, \pi]),$$

for any fixed $n \in \mathbb{Z}$, since θ is continuous on $[0, \pi]$ and

$$e^{i\theta(t)} = e^{i(t+2n\pi)} = e^{it}, \quad \text{for } t \in [0, \pi].$$

Note that these choices of continuous argument functions all differ by integer multiples of 2π.

Example 1.2

Determine a continuous argument function for the path

$$\Gamma : \gamma(t) = (2 + \sin t)e^{2it} \quad (t \in [0, 2\pi]),$$

illustrated in Figure 1.4.

Solution

Since $2 + \sin t > 0$, for $t \in [0, 2\pi]$, we have

$$|\gamma(t)| = |2 + \sin t| = 2 + \sin t, \quad \text{for } t \in [0, 2\pi],$$

so

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{2it}, \quad \text{for } t \in [0, 2\pi].$$

Thus one choice of continuous argument function is

$$\theta(t) = 2t \quad (t \in [0, 2\pi]).$$

Notice, in Example 1.2, that as t increases from 0 to 2π, the image point $\gamma(t)$ starts from $\gamma(0) = 2$ and returns to $\gamma(2\pi) = 2$, the same point; however, the value of the continuous argument function θ increases from $\theta(0) = 0$ to $\theta(2\pi) = 4\pi$, reflecting the fact that the path Γ winds twice around 0.

Exercise 1.2

Determine a continuous argument function for each of the following paths.

(a) $\Gamma : \gamma(t) = te^{\pi it} \quad (t \in [1, 3])$

(b) $\Gamma : \gamma(t) = 1 + it \quad (t \in [0, 1])$

(c) $\Gamma : \gamma(t) = e^{-4it} \quad (t \in [0, 2\pi])$

Intuitively it seems clear that every path Γ in $\mathbb{C} - \{0\}$ has a continuous argument function, and indeed this is so, as the following theorem confirms. However, the proof of this theorem is tricky (and is deferred until the end of this subsection) because, as you have seen earlier in the module, paths can be complicated.

Theorem 1.1

Any path $\Gamma : \gamma(t)$ $(t \in [a, b])$ lying in $\mathbb{C} - \{0\}$ has a continuous argument function θ, which is unique apart from the addition of a constant term of the form $2\pi n$, where $n \in \mathbb{Z}$.

The 'angular change' of $\gamma(t)$ along the path $\Gamma : \gamma(t)$ $(t \in [a, b])$ is given by $\theta(b) - \theta(a)$, so it follows that Γ winds around 0

$$\frac{1}{2\pi}(\theta(b) - \theta(a)) \text{ times}$$

(because an angular increase of 2π corresponds to one anticlockwise turn around 0). We use this quantity to define the *winding number* of any path (closed or not) around 0.

Definition

Let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a path lying in $\mathbb{C} - \{0\}$. The **winding number** of Γ around 0 is

$$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

where θ is any continuous argument function for Γ.

Remarks

1. The definition of $\operatorname{Wnd}(\Gamma, 0)$ does not depend on the choice of continuous argument function for Γ because (by Theorem 1.1) any two choices of continuous argument function for Γ differ by an integer multiple of 2π.

2. For an arbitrary path Γ, the winding number is a real number. However, for a closed path, the winding number is always an integer because $\theta(b)$ and $\theta(a)$ are both arguments of $\gamma(a) = \gamma(b)$, so they differ by an integer multiple of 2π. For example, the path

   $$\Gamma : \gamma(t) = 2e^{it} \quad (t \in [0, \pi])$$

   in Example 1.1, which is not closed, has continuous argument function $\theta(t) = t$ $(t \in [0, \pi])$, so

   $$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(\pi) - \theta(0)) = \frac{1}{2\pi}(\pi - 0) = \frac{1}{2}.$$

On the other hand, the path

$$\Gamma : \gamma(t) = (2 + \sin t)e^{2it} \quad (t \in [0, 2\pi])$$

in Example 1.2, which is closed, has continuous argument function $\theta(t) = 2t$ $(t \in [0, 2\pi])$, so

$$\mathrm{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(2\pi) - \theta(0)) = \frac{1}{2\pi}(4\pi - 0) = 2.$$

Most of the winding numbers that we consider will be of closed paths.

Exercise 1.3

Calculate the winding number around 0 of each of the paths Γ in Exercise 1.2 by using the continuous argument functions found in that exercise. Check your answers by sketching each of the paths Γ.

Exercise 1.4

Let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a path in $\mathbb{C} - \{0\}$, let c be in (a, b), and consider the subpaths $\Gamma_1 : \gamma(t)$ $(t \in [a, c])$ and $\Gamma_2 : \gamma(t)$ $(t \in [c, b])$. Prove that

$$\mathrm{Wnd}(\Gamma, 0) = \mathrm{Wnd}(\Gamma_1, 0) + \mathrm{Wnd}(\Gamma_2, 0).$$

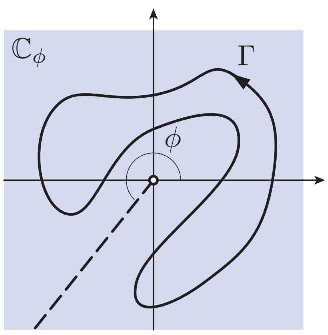


Figure 1.6 A path Γ in $\mathbb{C}_\phi$

It remains to prove Theorem 1.1. To do so, we make use of generalised argument functions, which were introduced in Subsection 5.1 of Unit C1. In Theorem 5.1 of Unit C1 we saw that the generalised argument function Arg_ϕ is continuous on the cut plane $\mathbb{C}_\phi$. It follows that if a path $\Gamma : \gamma(t)$ $(t \in [a, b])$ lies entirely in $\mathbb{C}_\phi$ (see Figure 1.6), then we can obtain a continuous argument function for Γ by choosing

$$\theta(t) = \mathrm{Arg}_\phi(\gamma(t)) \quad (t \in [a, b]).$$

So one way to obtain a continuous argument function for a more general path Γ is to break Γ up into subpaths, each of which lies in a cut plane, and then choose continuous argument functions for the subpaths, which match up at points where the subpaths join. This is the idea behind the proof of Theorem 1.1.

Proof of Theorem 1.1 Given any path $\Gamma : \gamma(t)$ $(t \in [a, b])$ lying in $\mathbb{C} - \{0\}$, we can apply the Paving Theorem (Theorem 3.7 of Unit B1) to partition the interval $[a, b]$ into adjacent subintervals $[t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n]$, with $t_0 = a$ and $t_n = b$, in such a way that

$$\gamma([t_{k-1}, t_k]) \subseteq D_k, \quad \text{for } k = 1, 2, \ldots, n,$$

where each D_k is an open disc in $\mathbb{C} - \{0\}$ (see Figure 1.7).

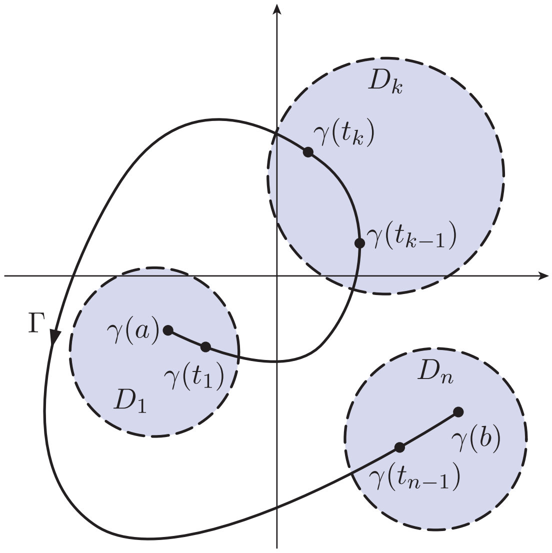


Figure 1.7 Paving the path Γ by discs D_k

If D_k has centre $r_k e^{i\theta_k}$, then $D_k \subseteq \mathbb{C}_{\phi_k}$, where $\phi_k = \theta_k + \pi$, so the argument function $\operatorname{Arg}_{\phi_k}$ is continuous on D_k. Thus, for $k = 1, 2, \ldots, n$, $\operatorname{Arg}_{\phi_k}(\gamma(t))$ is a continuous argument function for the subpath

$$\Gamma_k : \gamma(t) \quad (t \in [t_{k-1}, t_k]),$$

as illustrated in Figure 1.8.

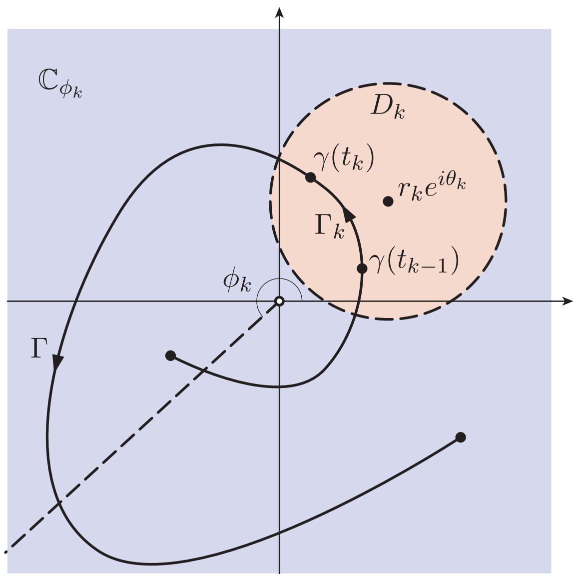


Figure 1.8 The subpath Γ_k of Γ lies inside the disc D_k

To ensure that these argument functions match up at the points $\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_n)$, we choose the arguments $\theta_1, \theta_2, \ldots, \theta_n$ (adding or subtracting integer multiples of 2π to each one in turn, as necessary) so that

$$\operatorname{Arg}_{\phi_{k+1}}(\gamma(t_k)) = \operatorname{Arg}_{\phi_k}(\gamma(t_k)), \quad \text{for } k = 1, 2, \ldots, n-1.$$

This is possible since $\gamma(t_k) \in D_k$ and $\gamma(t_k) \in D_{k+1}$.

It then follows that

$$\theta(t) = \begin{cases} \operatorname{Arg}_{\phi_1}(\gamma(t)), & t_0 \leq t \leq t_1, \\ \operatorname{Arg}_{\phi_2}(\gamma(t)), & t_1 \leq t \leq t_2, \\ \quad\vdots & \\ \operatorname{Arg}_{\phi_n}(\gamma(t)), & t_{n-1} \leq t \leq t_n, \end{cases}$$

is a continuous argument function for Γ. ■

1.2 The winding number as a contour integral

The winding number is a natural geometric concept, closely related to complex integration. The connection is made explicit in the following theorem, the proof of which uses generalised logarithm functions, which were introduced in Subsection 5.1 of Unit C1.

Theorem 1.2

Let $\Gamma : \gamma(t)$ $(t \in [a,b])$ be a closed contour lying in $\mathbb{C} - \{0\}$. Then

$$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z}\, dz.$$

Notice that Γ has to be a *contour* (a finite union of smooth paths), not just a path, for the integral to be defined.

Proof First we apply the Paving Theorem, as in the proof of Theorem 1.1, retaining the same notation. Since $D_k \subseteq \mathbb{C}_{\phi_k}$, we deduce that the function $f(z) = 1/z$ has a primitive $F(z) = \operatorname{Log}_{\phi_k}(z)$ on D_k, by Theorem 5.2 of Unit C1. Hence, by the Fundamental Theorem of Calculus (Theorem 3.1 of Unit B1),

$$\begin{aligned} \int_{\Gamma_k} \frac{1}{z}\, dz &= \operatorname{Log}_{\phi_k}(\gamma(t_k)) - \operatorname{Log}_{\phi_k}(\gamma(t_{k-1})) \\ &= \big(\log|\gamma(t_k)| + i \operatorname{Arg}_{\phi_k}(\gamma(t_k))\big) \\ &\quad - \big(\log|\gamma(t_{k-1})| + i \operatorname{Arg}_{\phi_k}(\gamma(t_{k-1}))\big) \\ &= \log|\gamma(t_k)| - \log|\gamma(t_{k-1})| + i(\theta(t_k) - \theta(t_{k-1})), \end{aligned}$$

for $k = 1, 2, \ldots, n$. Here θ denotes the continuous argument function for Γ that we obtained in the proof of Theorem 1.1. Then

$$\begin{aligned} \int_\Gamma \frac{1}{z}\, dz &= \sum_{k=1}^n \int_{\Gamma_k} \frac{1}{z}\, dz \\ &= \sum_{k=1}^n (\log|\gamma(t_k)| - \log|\gamma(t_{k-1})|) + i \sum_{k=1}^n (\theta(t_k) - \theta(t_{k-1})). \end{aligned}$$

Each of these two series is a 'telescoping series' in which terms cancel with one another to leave

$$\begin{aligned}\int_\Gamma \frac{1}{z}\,dz &= (\log|\gamma(t_n)| - \log|\gamma(t_0)|) + i(\theta(t_n) - \theta(t_0))\\ &= (\log|\gamma(b)| - \log|\gamma(a)|) + i(\theta(b) - \theta(a))\\ &= i(\theta(b) - \theta(a))\\ &= 2\pi i\,\mathrm{Wnd}(\Gamma, 0),\end{aligned}$$

where we have used the fact that $\gamma(a) = \gamma(b)$. Hence

$$\mathrm{Wnd}(\Gamma, 0) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z}\,dz,$$

as required. ■

For example, if Γ is the unit circle, then we know that

$$\int_\Gamma \frac{1}{z}\,dz = 2\pi i,$$

by Example 2.3 of Unit B1 (or by a simple application of the Residue Theorem). Consequently,

$$\mathrm{Wnd}(\Gamma, 0) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z}\,dz = 1,$$

which is as expected, because Γ winds once anticlockwise around the origin.

Although Theorem 1.2 can be useful for evaluating contour integrals of the form $\int_\Gamma 1/z\,dz$, its main use is to give certain other contour integrals a geometric interpretation (see, in particular, Section 2).

1.3 The winding number around an arbitrary point

We can define the winding number around an arbitrary point in a similar way to how we defined the winding number around 0.

Definitions

Let α be an arbitrary point in $\mathbb{C}$, and let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a path lying in $\mathbb{C} - \{\alpha\}$.

A **continuous argument function** for Γ relative to α is a continuous function $\theta_\alpha : [a, b] \longrightarrow \mathbb{R}$ such that, for each $t \in [a, b]$, $\theta_\alpha(t)$ is an argument of $\gamma(t) - \alpha$.

The **winding number** of Γ around α is

$$\mathrm{Wnd}(\Gamma, \alpha) = \frac{1}{2\pi}(\theta_\alpha(b) - \theta_\alpha(a)),$$

where θ_α is any continuous argument function for Γ relative to α.

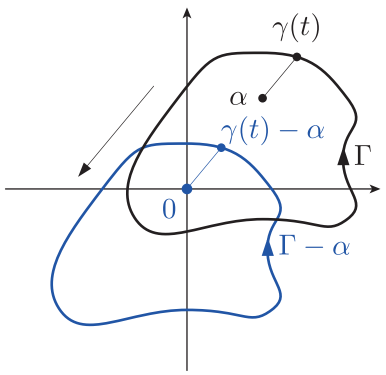


Figure 1.9 Translating Γ by $-\alpha$

There is an alternative way to define the winding number around α by using the translated path

$$\Gamma - \alpha : \gamma(t) - \alpha \quad (t \in [a, b]),$$

illustrated by Figure 1.9. We then define

$$\text{Wnd}(\Gamma, \alpha) = \text{Wnd}(\Gamma - \alpha, 0).$$

The two definitions of $\text{Wnd}(\Gamma, \alpha)$ are equivalent.

As before, the winding number around α can be determined by inspection if Γ is a given closed path that does not pass through α.

Exercise 1.5

Determine by inspection the winding number of the closed path

$$\Gamma : \gamma(t) = (2 + \sin t)e^{2it} \quad (t \in [0, 2\pi]),$$

illustrated in Figure 1.10, around each of the points 1, -2, $-2i$ and $3i$.

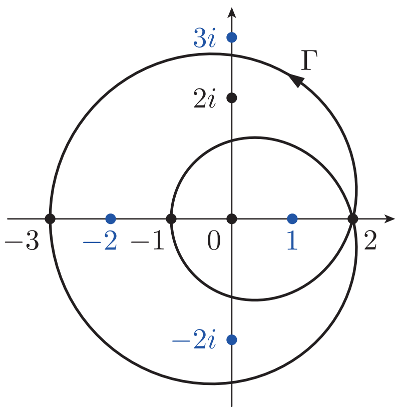


Figure 1.10 The closed path Γ

Exercise 1.6

Prove that if Γ is a closed contour and α does not lie on Γ, then

$$\text{Wnd}(\Gamma, \alpha) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - \alpha}\, dz.$$

(*Hint*: Use Theorem 1.2 and the equation $\text{Wnd}(\Gamma, \alpha) = \text{Wnd}(\Gamma - \alpha, 0)$.)

To finish this section we give a result about the variation of $\text{Wnd}(\Gamma, \alpha)$ as α varies. This result says that if Γ is a closed path, then the function $\alpha \longmapsto \text{Wnd}(\Gamma, \alpha)$ is constant on any open disc D lying in the complement of Γ, as illustrated in Figure 1.11.

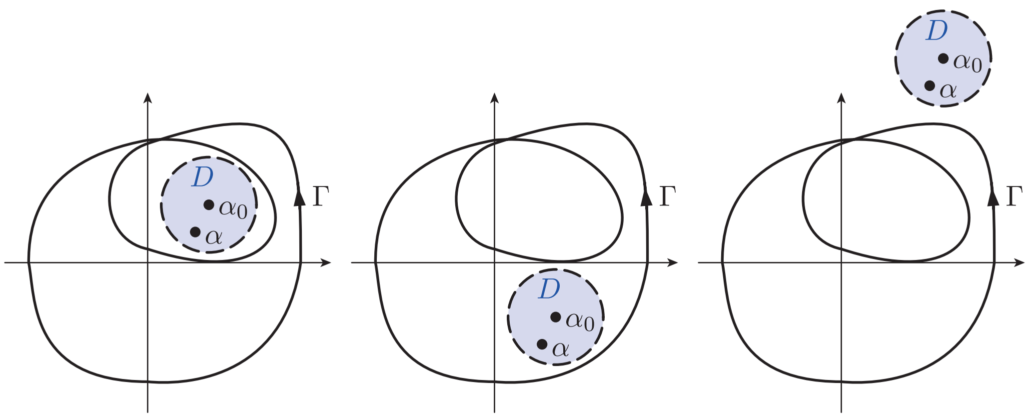


Figure 1.11 The winding numbers $\text{Wnd}(\Gamma, \alpha)$ and $\text{Wnd}(\Gamma, \alpha_0)$ are equal for each disc D in $\mathbb{C} - \Gamma$

Theorem 1.3

Let Γ be a closed path, and let D be an open disc in the complement of Γ. Then the function $\alpha \longmapsto \mathrm{Wnd}(\Gamma, \alpha)$ is constant on D.

Proof Suppose that θ_{α_0} is a continuous argument function for the path $\Gamma : \gamma(t)$ $(t \in [a, b])$ relative to the centre α_0 of D; that is, θ_{α_0} is continuous on $[a, b]$ and, for each $t \in [a, b]$, $\theta_{\alpha_0}(t)$ is an argument of $\gamma(t) - \alpha_0$.

We will use θ_{α_0} to construct a continuous argument function θ_α for Γ relative to α, where $\alpha \in D$. To do this, note that, for $t \in [a, b]$,

$$\gamma(t) - \alpha = (\gamma(t) - \alpha_0)\left(1 - \frac{\alpha - \alpha_0}{\gamma(t) - \alpha_0}\right). \tag{1.1}$$

Since Γ lies outside D, we have

$$|\alpha - \alpha_0| < |\gamma(t) - \alpha_0|, \quad \text{for } t \in [a, b],$$

so

$$\left|\frac{\alpha - \alpha_0}{\gamma(t) - \alpha_0}\right| < 1, \quad \text{for } t \in [a, b].$$

It follows that the point

$$z = 1 - \frac{\alpha - \alpha_0}{\gamma(t) - \alpha_0}$$

satisfies $|z - 1| < 1$, so this point lies in the cut plane $\mathbb{C}_\pi$ (see Figure 1.12). Thus the function

$$t \longmapsto \mathrm{Arg}\left(1 - \frac{\alpha - \alpha_0}{\gamma(t) - \alpha_0}\right) \quad (t \in [a, b])$$

is continuous.

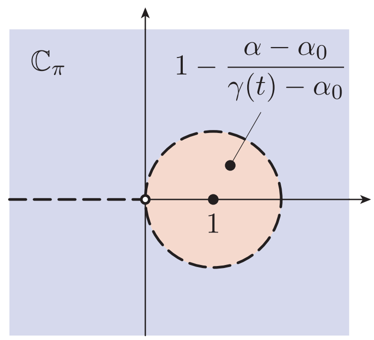


Figure 1.12 Point inside the disc $\{z : |z - 1| < 1\}$

Now, for each $t \in [a, b]$ we have that

$\theta_{\alpha_0}(t)$ is an argument of $\gamma(t) - \alpha_0$

and

$$\mathrm{Arg}\left(1 - \frac{\alpha - \alpha_0}{\gamma(t) - \alpha_0}\right) \text{ is an argument of } 1 - \frac{\alpha - \alpha_0}{\gamma(t) - \alpha_0}.$$

Since the sum of arguments of two complex numbers is an argument of the product of the two complex numbers (Subsection 2.3 of Unit A1), we see from equation (1.1) that

$$\theta_{\alpha_0}(t) + \mathrm{Arg}\left(1 - \frac{\alpha - \alpha_0}{\gamma(t) - \alpha_0}\right)$$

is an argument of $\gamma(t) - \alpha$. Therefore

$$\theta_\alpha(t) = \theta_{\alpha_0}(t) + \mathrm{Arg}\left(1 - \frac{\alpha - \alpha_0}{\gamma(t) - \alpha_0}\right) \quad (t \in [a, b]) \tag{1.2}$$

is a continuous argument function for Γ relative to α.

Since $\gamma(b) = \gamma(a)$, we have

$$\operatorname{Arg}\left(1 - \frac{\alpha - \alpha_0}{\gamma(b) - \alpha_0}\right) = \operatorname{Arg}\left(1 - \frac{\alpha - \alpha_0}{\gamma(a) - \alpha_0}\right),$$

so $\theta_\alpha(b) - \theta_{\alpha_0}(b) = \theta_\alpha(a) - \theta_{\alpha_0}(a)$, by equation (1.2). Hence

$$\theta_\alpha(b) - \theta_\alpha(a) = \theta_{\alpha_0}(b) - \theta_{\alpha_0}(a),$$

which implies that

$$\operatorname{Wnd}(\Gamma, \alpha) = \operatorname{Wnd}(\Gamma, \alpha_0),$$

as required. ■

Further exercises

Exercise 1.7

The closed path

$$\Gamma : \gamma(t) = \left(2 + \cos \tfrac{3}{2}t\right)e^{it} \quad (t \in [0, 4\pi])$$

is shown in Figure 1.13.

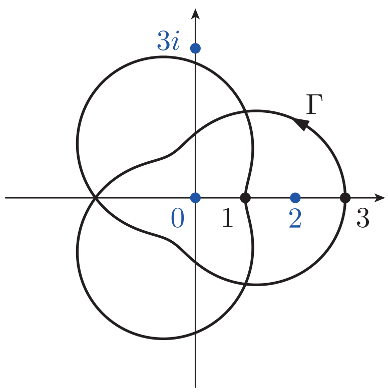


Figure 1.13 The closed path Γ

(a) Determine by inspection the winding number of Γ around each of the points 0, 2 and $3i$.

(b) Determine a continuous argument function for Γ, and hence verify the value of $\operatorname{Wnd}(\Gamma, 0)$ that you found in part (a).

(c) Use the result of Exercise 1.6 to evaluate

$$\int_\Gamma \frac{1}{z-2}\, dz.$$

Exercise 1.8

Determine a continuous argument function for the path

$$\Gamma : \gamma(t) = -1 + it \quad (t \in [-1, 1]),$$

and hence evaluate $\operatorname{Wnd}(\Gamma, 0)$.

(*Hint*: Note that $\Gamma \subseteq \mathbb{C}_{2\pi}$.)

2 Locating zeros of analytic functions

After working through this section, you should be able to:

- understand and apply the Argument Principle
- understand Rouché's Theorem and use it to determine the number of zeros of an analytic function in a specified region.

2.1 The Argument Principle

We now begin to investigate the location of zeros of an analytic function. Recall from Subsection 5.1 of Unit B3 that a function f that is analytic at a point α has a *zero of order* k *at* α if

$$f(\alpha) = f^{(1)}(\alpha) = f^{(2)}(\alpha) = \cdots = f^{(k-1)}(\alpha) = 0, \quad \text{but } f^{(k)}(\alpha) \neq 0.$$

Recall also that if f is analytic and not identically zero on a region, then any zero of f on that region is of finite order, and each zero is isolated (Theorems 5.2 and 5.3 of Unit B3).

The key to the main result of this subsection – the Argument Principle – is to relate the zeros of an analytic function f to the poles of the function f'/f. This function is called the **logarithmic derivative** of f since

$$\frac{d}{dz}\operatorname{Log}(f(z)) = \frac{f'(z)}{f(z)}.$$

Example 2.1

Let $f(z) = (z-i)^2(z+1)^3$. Determine the poles of the function f'/f and relate them to the zeros of f.

Solution

Since

$$f'(z) = 2(z-i)(z+1)^3 + (z-i)^2 \times 3(z+1)^2,$$

we have

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{2(z-i)(z+1)^3 + 3(z-i)^2(z+1)^2}{(z-i)^2(z+1)^3} \\ &= \frac{2}{z-i} + \frac{3}{z+1}.\end{aligned}$$

Thus f'/f has a simple pole at i with residue 2, and it has a simple pole at -1 with residue 3.

In comparison, f has a zero at i of order 2, and it has a zero at -1 of order 3.

Exercise 2.1

Let $f(z) = z^{10}(z-1)$. Determine the poles of the function f'/f and relate them to the zeros of f.

The solutions to Example 2.1 and Exercise 2.1 suggest a connection between the order of a zero of f and the residue at the corresponding pole of f'/f. We now establish this connection.

Theorem 2.1

Let f be an analytic function with a zero of order n at α. Then the function f'/f has a simple pole at α with

$$\operatorname{Res}(f'/f, \alpha) = n.$$

Proof By assumption

$$f(z) = (z-\alpha)^n g(z),$$

where g is a function that is analytic on some open disc D with centre α, and $g(\alpha) \neq 0$ (Theorem 5.1 of Unit B3). Therefore

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{n(z-\alpha)^{n-1}g(z) + (z-\alpha)^n g'(z)}{(z-\alpha)^n g(z)} \\ &= \frac{n}{z-\alpha} + \frac{g'(z)}{g(z)}.\end{aligned}$$

Since g is analytic at α and $g(\alpha) \neq 0$, it follows that g'/g is analytic at α. Hence f'/f has a simple pole at α with residue n, as required. ■

Using Theorem 2.1, we can prove the following theorem, which gives an integral formula for the number of zeros of an analytic function f inside a simple-closed contour.

Theorem 2.2

Let f be a function that is analytic on a simply connected region $\mathcal{R}$, and let Γ be a simple-closed contour in $\mathcal{R}$ such that $f(z) \neq 0$, for $z \in \Gamma$. Then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz$$

is equal to the number of zeros of f inside Γ, counted according to their orders.

Remark

The phrase 'counted according to their orders' means that, for example, if the zeros of f inside Γ are α_1, α_2 and α_3, and the orders of these zeros are 1, 3 and 2, respectively, then

the number of zeros of f inside Γ is $1 + 3 + 2 = 6$.

Some texts use the word *multiplicity* instead of order.

Proof By the Residue Theorem,

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz$$

is equal to the sum of the residues of f'/f at the singularities of f'/f lying inside Γ. Since f' is analytic (because f is), f'/f is analytic at any point of $\mathcal{R}$ where f is non-zero. Hence, by Theorem 2.1, the only singularities of f'/f in $\mathcal{R}$ are simple poles at the zeros of f. Since f is not identically zero on $\mathcal{R}$, we see from Theorem 5.4 of Unit B3 that the set S of zeros of f inside Γ does not have a limit point in $\mathcal{R}$. Hence S comprises only finitely many points $\alpha_1, \alpha_2, \ldots, \alpha_k$, say, of orders $n_1, n_2, \ldots, n_k$, respectively (see Figure 2.1). Then, by Theorem 2.1, $\operatorname{Res}(f'/f, \alpha_k) = n_k$, for $n = 1, 2, \ldots, k$, so

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz = n_1 + n_2 + \cdots + n_k,$$

which is the number of zeros of f inside Γ, counted according to their orders. ■

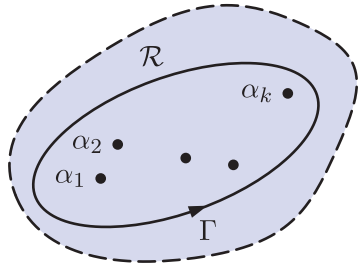


Figure 2.1 Zeros $\alpha_1, \alpha_2, \ldots, \alpha_k$ of f inside Γ

For example, if $f(z) = z^3$ and Γ is the unit circle $\{z : |z| = 1\}$, then

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi i}\int_\Gamma \frac{3z^2}{z^3}\,dz\\ &= \frac{3}{2\pi i}\int_\Gamma \frac{1}{z}\,dz = 3.\end{aligned}$$

It follows from Theorem 2.2 that the number of zeros of the function $f(z) = z^3$ lying inside Γ is three, counted according to their orders; this is as expected, since f has a single zero inside Γ, namely the zero of order three at 0.

The importance of Theorem 2.2 lies in the geometric interpretation of the integral of f'/f along Γ, which turns out to be the winding number of the image path $f(\Gamma)$ around 0. Thus we obtain the following result.

Theorem 2.3 Argument Principle

Let f be a function that is analytic on a simply connected region $\mathcal{R}$, and let Γ be a simple-closed contour in $\mathcal{R}$ such that $f(z) \neq 0$, for $z \in \Gamma$. Then

$$\operatorname{Wnd}(f(\Gamma), 0)$$

is equal to the number of zeros of f inside Γ, counted according to their orders.

Proof By Theorem 2.2, it is sufficient to prove that $\operatorname{Wnd}(f(\Gamma), 0)$ is equal to $\dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f'(z)}{f(z)}\,dz$. To do this, let us first assume that Γ has a smooth parametrisation $\gamma\colon [a,b] \longrightarrow \mathbb{C}$. Then

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\,\gamma'(t)\,dt\\ &= \frac{1}{2\pi i}\int_a^b \frac{1}{(f\circ\gamma)(t)}\,(f\circ\gamma)'(t)\,dt \quad \text{(by the Chain Rule)}\\ &= \frac{1}{2\pi i}\int_{f(\Gamma)} \frac{1}{w}\,dw\\ &= \operatorname{Wnd}(f(\Gamma), 0),\end{aligned}$$

since $t \longmapsto (f\circ\gamma)(t)$ $(t \in [a,b])$ is the parametrisation of $f(\Gamma)$.

The more general case when Γ is a contour, not necessarily a smooth path, then follows by splitting Γ into its constituent smooth paths

$$\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n,$$

as in Unit B1, before applying the reasoning above.

Actually, the reasoning is valid only if f' is non-zero on Γ, for then the image of each constituent smooth path of Γ is a constituent smooth path of $f(\Gamma)$, so $f(\Gamma)$ is a contour. In general, however, f' can have zeros on Γ, but only finitely many. Thus, by making small modifications to Γ near the zeros of f' (see Figure 2.2), we can obtain a simple-closed contour Γ' on which f' does *not* vanish and for which

- Γ' surrounds the same zeros of f as does Γ
- $\operatorname{Wnd}(f(\Gamma'), 0) = \operatorname{Wnd}(f(\Gamma), 0)$.

Since the theorem holds for Γ', it must also hold for Γ. ■

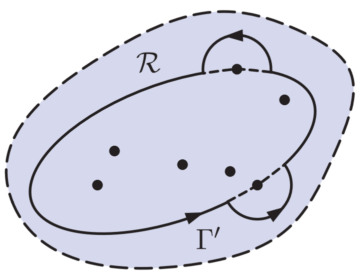


Figure 2.2 Avoiding the zeros of f' on Γ

By applying the Argument Principle to the function $f - \beta\colon z \longmapsto f(z) - \beta$ instead of f, we obtain this useful corollary.

Corollary

Let f be a function that is analytic on a simply connected region $\mathcal{R}$, and let Γ be a simple-closed contour in $\mathcal{R}$ such that $f(z) \neq \beta$, for $z \in \Gamma$. Then $\operatorname{Wnd}(f(\Gamma), \beta)$ is the number of zeros of the function $f - \beta$ inside Γ, counted according to their orders.

Remark

Each zero of $f - \beta$ is a solution of the equation $f(z) = \beta$.

As an example, consider the polynomial function $f(z) = z^3 - z^2$ and the closed contour $\Gamma = \{z : |z| = 2\}$. The image contour $f(\Gamma)$ is shown in Figure 2.3.

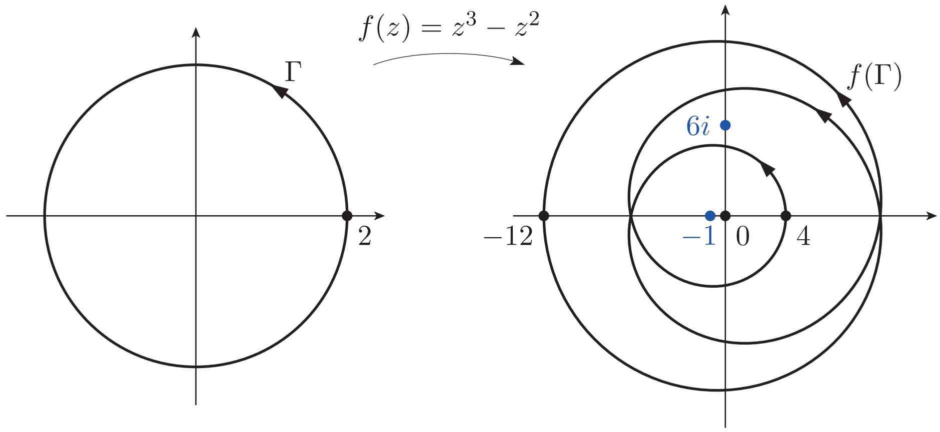


Figure 2.3 Image of $\Gamma = \{z : |z| = 2\}$ under $f(z) = z^3 - z^2$

Notice first that $f(z) \neq 0$ for $z \in \Gamma$, and that $f(\Gamma)$ winds three times around 0, which implies, by Theorem 2.3, that f has three zeros inside Γ (where, as usual, we count zeros according to their orders). This is no surprise, since

$$\begin{aligned} f(z) = 0 &\iff z^3 - z^2 = 0 \\ &\iff z^2(z-1) = 0, \end{aligned}$$

so f has a simple zero at 1 and a zero of order two at 0, both inside Γ.

A less obvious result is obtained by noting that $f(\Gamma)$ winds twice around the point $6i$ (see Figure 2.3), so, by the corollary, f takes the value $6i$ twice inside Γ. This implies that the equation $f(z) = 6i$ (that is, $z^3 - z^2 = 6i$) has two solutions inside Γ, which is not at all obvious.

Exercise 2.2

Let $f(z) = z^3 - z^2$ and $\Gamma = \{z : |z| = 2\}$. Use Figure 2.3 to determine the number of solutions of the equation $f(z) = -1$ inside Γ.

2.2 Rouché's Theorem

In the previous subsection we saw that an accurate diagram of the image $f(\Gamma)$ of a simple-closed contour Γ under an analytic function f gives a great deal of information about solutions of equations of the form $f(z) = \beta$. Since such a diagram is not always easy to obtain, it is useful to have other methods to give information about zeros of functions.

This subsection is about one such method, based on a result called Rouché's Theorem (where Rouché is pronounced 'roo-shay'). To prepare for this, we first consider an example that is closely related to Exercise 2.2.

Figure 2.4 shows the image of the circle $\Gamma = \{z : |z| = 2\}$ under the function $f(z) = z^3 - z^2 + 1$.

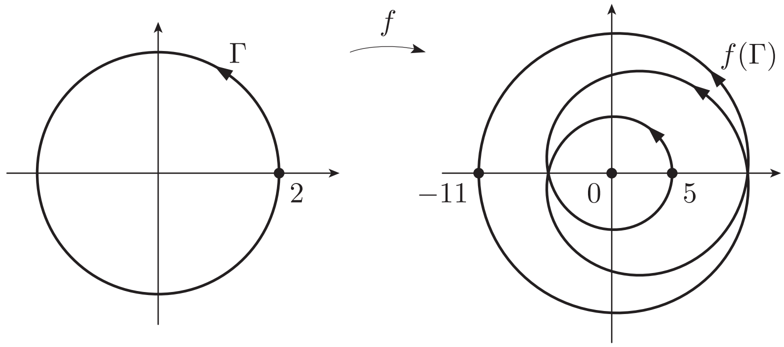


Figure 2.4 Image of $\Gamma = \{z : |z| = 2\}$ under $f(z) = z^3 - z^2 + 1$

Since the image path $f(\Gamma)$ winds around 0 three times anticlockwise, we deduce that $\mathrm{Wnd}(f(\Gamma), 0) = 3$, so f has three zeros in $\{z : |z| < 2\}$.

It would be desirable to be able to reach this observation without sketching $f(\Gamma)$. We can do so by observing that f behaves in a similar way to the simpler function $g(z) = z^3$. Since the image path $g(\Gamma)$ is the circle of radius 8 traversed three times anticlockwise, it follows that $\mathrm{Wnd}(g(\Gamma), 0) = 3$. Our strategy is to show that the paths $f(\Gamma)$ and $g(\Gamma)$ are sufficiently close that their winding numbers around 0 must coincide.

To this end, observe that, for $z \in \Gamma$,

$$|f(z) - g(z)| = \left|-z^2 + 1\right| \leq |z|^2 + 1 = 5,$$

by the Triangle Inequality. Thus if z is a point on Γ, then $f(z)$ belongs to the closed disc of radius 5 centred at $g(z)$, as shown in Figure 2.5. This disc misses the origin, because $|g(z)| = 8$.

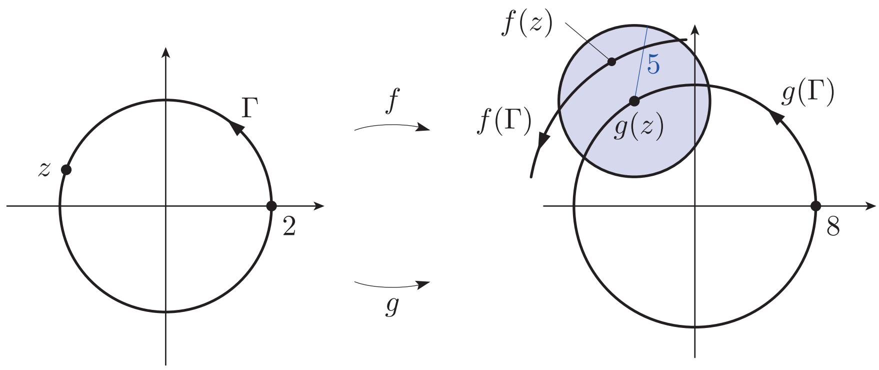


Figure 2.5 The distance between the image points $f(z)$ and $g(z)$ is at most 5

As a point z traverses Γ, the image point $g(z)$ traverses $g(\Gamma)$, taking with it the disc and point $f(z)$ inside. Thus both paths $g(\Gamma)$ and $f(\Gamma)$ wind around the origin the same number of times; that is,

$$\mathrm{Wnd}(f(\Gamma), 0) = \mathrm{Wnd}(g(\Gamma), 0) = 3,$$

so f has three zeros in Γ. It is as if $g(z)$ goes for a walk around the origin (three times) with a dog $f(z)$ on a lead, and the dog is never allowed to get near the origin. The dog $f(z)$ also ends up walking around the origin three times, even though its path is quite different to that of $g(z)$.

Motivated by the discussion above, we now state Rouché's Theorem for counting zeros of analytic functions.

Theorem 2.4 Rouché's Theorem

Suppose that f and g are analytic functions on a simply connected region $\mathcal{R}$, and Γ is a simple-closed contour in $\mathcal{R}$ with

$$|f(z) - g(z)| < |g(z)|, \quad \text{for } z \in \Gamma.$$

Then f has the same number of zeros as g inside Γ, each counted according to their orders.

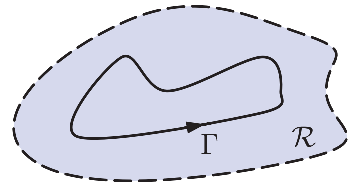


Figure 2.6 A simple-closed contour Γ in a simply connected region $\mathcal{R}$

The region $\mathcal{R}$ and path Γ are illustrated in Figure 2.6.

Proof If $z \in \Gamma$, then $|f(z) - g(z)| < |g(z)|$. Let us use this inequality to show that neither $f(z)$ nor $g(z)$ is 0, for $z \in \Gamma$. If $f(z) = 0$, then $|0 - g(z)| < |g(z)|$, which is impossible, and if $g(z) = 0$, then $|f(z) - 0| < 0$, which is also impossible. Therefore $f(z) \neq 0$ and $g(z) \neq 0$, for $z \in \Gamma$.

Now let $h(z) = f(z)/g(z)$. This function is analytic on $\mathcal{R}$ apart from isolated singularities at the zeros of g. The observation from the previous paragraph tells us that h has no singularities or zeros on Γ.

Notice that

$$|h(z) - 1| = \frac{|f(z) - g(z)|}{|g(z)|} < 1, \quad \text{for } z \in \Gamma,$$

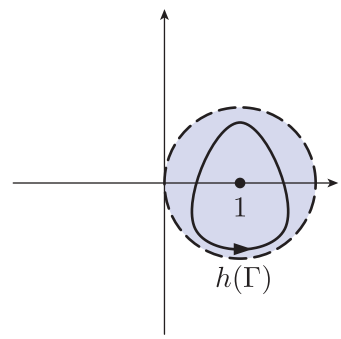


Figure 2.7 The path $h(\Gamma)$ lies inside the disc $\{z : |z - 1| < 1\}$

which implies that $h(\Gamma)$ lies wholly within the disc centred at 1 of radius 1 (see Figure 2.7). Therefore $\operatorname{Wnd}(h(\Gamma), 0) = 0$. Hence, by Theorems 2.2 and 2.3,

$$\frac{1}{2\pi i} \int_\Gamma \frac{h'(z)}{h(z)}\, dz = \operatorname{Wnd}(h(\Gamma), 0) = 0.$$

Next, since $f(z) = g(z)h(z)$, we obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)}\, dz &= \frac{1}{2\pi i} \int_\Gamma \frac{g'(z)h(z) + g(z)h'(z)}{g(z)h(z)}\, dz \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{g'(z)}{g(z)}\, dz + \frac{1}{2\pi i} \int_\Gamma \frac{h'(z)}{h(z)}\, dz. \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{g'(z)}{g(z)}\, dz. \end{aligned}$$

Therefore the number of zeros of f inside Γ is equal to the number of zeros of g inside Γ, by Theorem 2.2. ■

The function g in Rouché's Theorem is referred to as a **dominant term** for f on Γ. For example, in the discussion preceding the theorem, we considered the function $f(z) = z^3 - z^2 + 1$ and used the dominant term $g(z) = z^3$. This function g was chosen as the dominant term because it is simpler than f (we know precisely what the zeros of g are) and because it satisfies the inequality $|f(z) - g(z)| < |g(z)|$ on Γ. In general, the choice of g is not unique (as we will see in the solution to Exercise 2.8(a)(ii), for example), but often the form of $f(z)$ suggests an obvious choice for g.

Example 2.2

Determine how many zeros of the function

$$f(z) = e^z + 3z^2$$

lie in the disc $\{z : |z| < 1\}$.

Solution

On $\Gamma = \{z : |z| = 1\}$ a dominant term for f is $g(z) = 3z^2$, since, for $z \in \Gamma$,

$$|f(z) - g(z)| = |e^z| = e^{\operatorname{Re} z} \leq e^1 = e$$

and $|g(z)| = 3 > e$.

Now, g has a zero of order two at 0, which is in $\{z : |z| < 1\}$ (and it has no other zeros). Since f and g are analytic on the simply connected region $\mathbb{C}$, and Γ is a simple-closed contour in $\mathbb{C}$, we can apply Rouché's Theorem to see that f has two zeros in $\{z : |z| < 1\}$.

Exercise 2.3

(a) Determine how many zeros of the function

$$f(z) = z^5 + z^3 + iz + 1$$

lie in the disc $\{z : |z| < 2\}$.

(b) Determine how many zeros of the function

$$f(z) = e^z - \tfrac{1}{3}z^4$$

lie in the disc $\{z : |z| < 1\}$.

(c) Determine how many zeros of the function

$$f(z) = z^5 - 3z^3 - 1$$

lie in each of the following regions.

(i) $\{z : |z| < 2\}$ (ii) $\{z : |z| < 1\}$ (iii) $\{z : 1 < |z| < 2\}$

Exercise 2.4

(a) Use the Taylor series about 0 for exp to prove that

$$|e^z - 1| \le e - 1, \quad \text{for } |z| \le 1.$$

(b) Use the inequality in part (a) to prove that the equation

$$e^z = 2z + 1$$

has exactly one solution in the disc $\{z : |z| < 1\}$.

Remarks

1. In some texts the statement of Rouché's Theorem is formulated in a slightly different way to Theorem 2.4, with the key inequality

   $$|f(z) - g(z)| < |g(z)|, \quad \text{for } z \in \Gamma,$$

   replaced by

   $$|g(z)| < |f(z)|, \quad \text{for } z \in \Gamma.$$

   This amounts to replacing g by f and f by $f + g$, so the conclusion with this alternative formulation of the theorem is that f has the same number of zeros as $f + g$ inside Γ. We never use this alternative formulation.

2. In Exercise 2.3(b) you should have found that the function $f(z) = e^z - \frac{1}{3}z^4$ has no zeros in the disc $\{z : |z| < 1\}$. Sometimes when proving that a function has no zeros you can apply more elementary arguments than Rouché's Theorem. In this case, for $|z| \le 1$, we have

   $$|e^z| = e^{\operatorname{Re} z} \ge e^{-1},$$

   so

   $$|f(z)| = |e^z - \tfrac{1}{3}z^4| \ge |e^z| - \tfrac{1}{3}|z^4| \ge e^{-1} - \tfrac{1}{3} > 0,$$

   by the backwards form of the Triangle Inequality. Hence f has no zeros in $\{z : |z| \le 1\}$.

In the examples of applying Rouché's Theorem so far, the simple-closed contour Γ has always been a circle, and the main problem was to choose a suitable dominant term g on Γ. In the next example we must first choose an appropriate simple-closed contour before we can apply Rouché's Theorem.

Example 2.3

Prove that the equation

$$z + e^{-z} = 2$$

has exactly one solution in the right half-plane $\{z : \operatorname{Re} z > 0\}$.

Solution

First note that the equation $z + e^{-z} = 2$ can be rewritten in the form $f(z) = 0$, where

$$f(z) = z + e^{-z} - 2.$$

Next notice that the boundary of the right half-plane is not a simple-closed contour, so neither Rouché's Theorem nor the Argument Principle can be applied directly to this set. Instead we consider open semi-discs of the form

$$S = \{z : |z| < r,\ \operatorname{Re} z > 0\},$$

and prove that, for any $r > 3$, f has exactly one zero in S. It then follows that f has exactly one zero in $\{z : \operatorname{Re} z > 0\}$, as required.

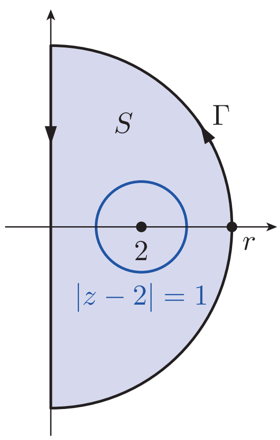


Figure 2.8 The circle $\{z : |z-2| = 1\}$ inside the open semi-disc S

We let Γ be the semicircular simple-closed contour that forms the boundary of S, traversed anticlockwise, and we try to choose a dominant term for f on Γ. To do this it helps to notice that

$$|e^{-z}| = e^{-\operatorname{Re} z} \leq 1, \quad \text{for } z \in \Gamma, \tag{2.1}$$

since $\operatorname{Re} z \geq 0$, for $z \in \Gamma$. On the other hand, if $r > 3$, then

$$|z-2| > 1, \quad \text{for } z \in \Gamma, \tag{2.2}$$

since $\{z : |z-2| = 1\}$ lies inside Γ when $r > 3$ (see Figure 2.8).

Combining inequalities (2.1) and (2.2), we obtain

$$|e^{-z}| < |z-2|, \quad \text{for } z \in \Gamma.$$

Let $g(z) = z - 2$. Then $f(z) - g(z) = e^{-z}$, so g is a dominant term for f on Γ. Since f and g are entire, we can apply Rouché's Theorem to deduce that f has the same number of zeros as g inside Γ, namely one ($g(z) = 0 \iff z = 2$). Hence f has exactly one zero in any such semi-disc S, so it has exactly one zero in $\{z : \operatorname{Re} z > 0\}$, as required.

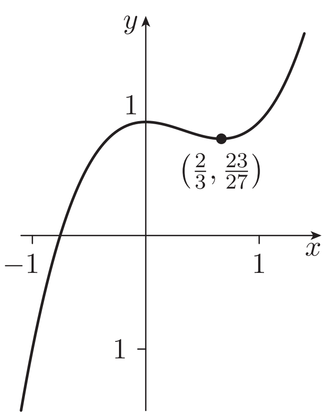


Figure 2.9 Graph of $f(x) = x^3 - x^2 + 1$

We can often be more precise about the locations of the zeros of a given analytic function f by studying the restriction of f to a particular set, such as the real axis. For example, at the start of this subsection we saw that the function $f(z) = z^3 - z^2 + 1$ has three zeros inside the circle $\Gamma = \{z : |z| = 2\}$. By sketching the graph of the real function $x \longmapsto f(x)$ using calculus (see Figure 2.9), we see that f has exactly one real zero, which lies in the interval $(-1, 0)$, so it is inside Γ. Since f is a real polynomial function, it follows (from Exercise 3.9 of Unit A1) that the other two zeros of f are complex conjugates of one another. Indeed, if

$$f(z) = z^3 - z^2 + 1 = 0,$$

then

$$f(\overline{z}) = \overline{z}^3 - \overline{z}^2 + 1 = \overline{z^3 - z^2 + 1} = 0.$$

A similar argument can be applied whenever the function f satisfies

$$\overline{f(z)} = f(\overline{z}).$$

For instance,

$$\overline{\exp(z)} = \exp(\overline{z}) \quad \text{and} \quad \overline{\sin z} = \sin \overline{z},$$

by Exercise 4.11 of Unit A2.

Exercise 2.5

Prove that the two zeros of the function $f(z) = e^z + 3z^2$ in the disc $\{z : |z| < 1\}$ (see Example 2.2) form a pair of complex conjugates.

Remark

The same kind of argument cannot be used with a function such as

$$f(z) = z^5 + z^3 + iz + 1,$$

since the coefficients of this polynomial are not all real.

Further exercises

Exercise 2.6

Let $f(z) = (z-1)^3(z-2)^2(z-3)$.

(a) Determine the poles of the function f'/f and find the residues of f'/f.

(b) Let $\Gamma = \{z : |z| = 4\}$. Determine $\operatorname{Wnd}(f(\Gamma), 0)$.

Exercise 2.7

Let $f(z) = z - z^2$ and $\Gamma = \{z : |z| = 1\}$. Use the diagram of the image path $f(\Gamma)$ in Figure 2.10 to determine the number of solutions inside Γ for each of the following equations.

(a) $f(z) = -\frac{1}{2}$ (b) $f(z) = \frac{1}{4}$ (c) $f(z) = 2i$

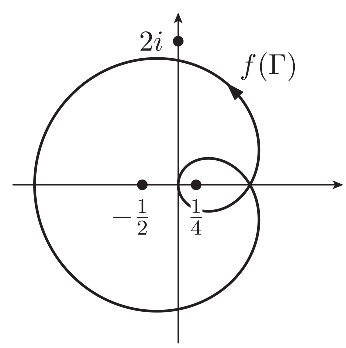


Figure 2.10 The image path $f(\Gamma)$

Exercise 2.8

For each of the following functions f, determine the number of zeros of f in the given regions.

(a) $f(z) = z^5 + 3z + 10$

(i) $S_1 = \{z : |z| < 2\}$ (ii) $S_2 = \{z : |z| < 1\}$

(iii) $S_3 = \{z : 1 < |z| < 2\}$ (iv) $S_4 = \{z : \operatorname{Im} z > 0\}$

(b) $f(z) = 3z + \operatorname{Log}(1+z)$, $S = \left\{z : |z| < \frac{1}{2}\right\}$

(*Hint*: In part (b) use the inequality $|\operatorname{Log}(1+z)| \le 2|z|$, for $|z| \le \frac{1}{2}$.)

History of Rouché's Theorem

In 1862, after the death of Cauchy (in 1857), the French mathematician Eugène Rouché (1832–1910) published a memoir on the work of Lagrange, which included revised versions of a number of Cauchy's results. Rouché felt that Cauchy's work was obscure and difficult to read; he wrote that the theorems of Cauchy were

> presented in a more or less explicit manner in the middle of a great apparatus of formulae and complicated notation.
>
> (Bottazzini and Gray, 2013, p. 210)

Found at the end of Rouché's paper, as an afterthought, is the theorem now known as Rouché's Theorem, stated in the special case of a circular path. However, even this result was known to Cauchy – it can be found in a more general form in one of his papers from 1831.

3 Local behaviour of analytic functions

After working through this section, you should be able to:

- state the Open Mapping Theorem and appreciate its application to regions
- state the Local Mapping Theorem and use it to determine the local behaviour of an analytic function
- state and use the Inverse Function Rule
- obtain the Taylor series of the inverse function of a given one-to-one analytic function.

3.1 The Open Mapping Theorem

We have seen that analytic functions have many remarkable properties and that many formulas hold for analytic functions that do not hold for general complex functions. Using the Argument Principle, we now demonstrate another important property of analytic functions.

Theorem 3.1 Open Mapping Theorem

Let f be a function that is analytic and non-constant on a region $\mathcal{R}$, and let G be an open subset of $\mathcal{R}$. Then $f(G)$ is open.

This theorem will be proved in Subsection 3.3.

We observe that if f is a constant function, then $f(G)$ is a set comprising a single element, so $f(G)$ is certainly not open.

At first sight the Open Mapping Theorem may not seem particularly remarkable. However, many complex functions do not map open sets to open sets. For example, the function

$$f(z) = f(x+iy) = x^2 + iy^2,$$

which is not analytic on $\mathbb{C}$ (since the Cauchy–Riemann equations are satisfied only when $y = x$), maps the open set $\mathbb{C}$ to the closed upper-right quadrant $\{u + iv : u \geq 0,\ v \geq 0\}$, which is not an open set (see Figure 3.1).

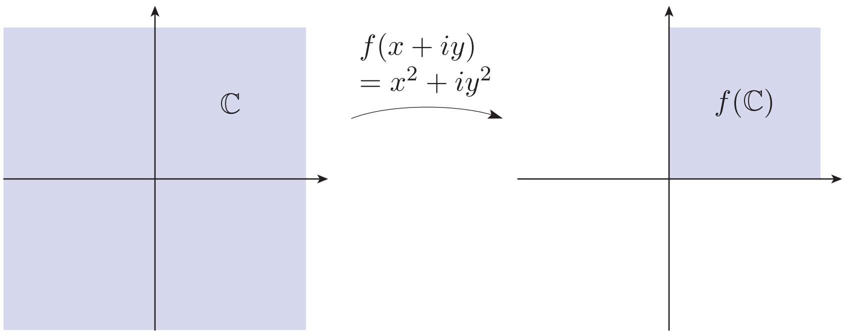


Figure 3.1 Image of $\mathbb{C}$ under $f(x+iy) = x^2 + iy^2$

The Open Mapping Theorem provides an easy way of proving that there are no analytic functions with certain properties.

Exercise 3.1

Prove that there are no non-constant entire functions that map $\mathbb{C}$ into $\mathbb{R}$.

One immediate consequence of the Open Mapping Theorem is that non-constant analytic functions map regions to regions.

Corollary

Let f be a function that is analytic and non-constant on a region $\mathcal{R}$. Then $f(\mathcal{R})$ is also a region.

Proof A region $\mathcal{R}$ is a connected open set. The Open Mapping Theorem shows that $f(\mathcal{R})$ is open, and Theorem 4.2 of Unit A3 shows that $f(\mathcal{R})$ is connected. Hence $f(\mathcal{R})$ is also a region. ■

Next we describe a result, closely related to the Open Mapping Theorem, which gives a description of the different types of *local* behaviour of an analytic function. First we look at a simple example.

Example 3.1

Let $f(z) = z^3$.

(a) Prove that for each non-zero complex number w the equation $f(z) = w$ has three distinct solutions.

(b) Determine an open disc D with centre 1 such that the restriction of the function f to D is one-to-one.

(c) Explain why it is impossible to find an open disc D with centre 0 such that the restriction of f to D is one-to-one.

Solution

(a) If $w \neq 0$, then w can be written in the form

$$w = \rho(\cos\phi + i\sin\phi), \quad \rho > 0,\ -\pi < \phi \leq \pi.$$

Then, by Theorem 3.1 of Unit A1, the equation $w = z^3$ has exactly three distinct solutions

$$z_k = \rho^{1/3}\left(\cos\left(\tfrac{1}{3}\phi + \tfrac{2}{3}k\pi\right) + i\sin\left(\tfrac{1}{3}\phi + \tfrac{2}{3}k\pi\right)\right), \quad k = 0, 1, 2,$$

as illustrated in Figure 3.2.

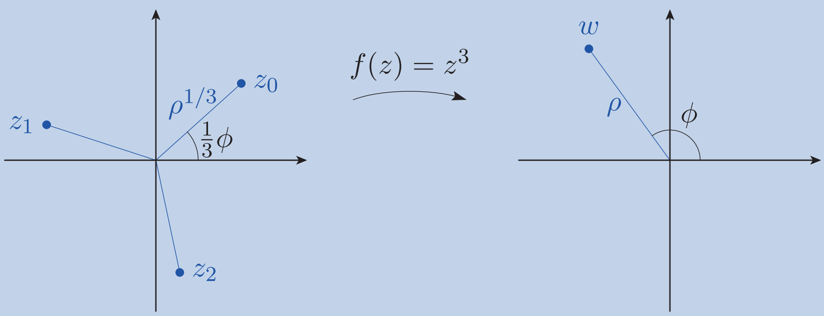


Figure 3.2 The points z_0, z_1 and z_2 each map to w under $f(z) = z^3$

(b) From part (a) we see that the restriction of f to the sector

$$A = \{z : |\operatorname{Arg} z| < \pi/3\}$$

is one-to-one, because, for each non-zero complex number w, there is at most one complex number z in A with $z^3 = w$.

Using Figure 3.3 and the formula $\sin\pi/3 = \sqrt{3}/2$, we can see that the open disc $D = \left\{z : |z - 1| < \sqrt{3}/2\right\}$ lies in A. It follows that the restriction of f to D is one-to-one.

(c) Suppose that D is an open disc with centre 0. If $z \in D$, then

$$f\left(ze^{2\pi i/3}\right) = \left(ze^{2\pi i/3}\right)^3 = z^3 = f(z).$$

The points z and $ze^{2\pi i/3}$ are different, for $z \neq 0$, so f is not one-to-one on D.

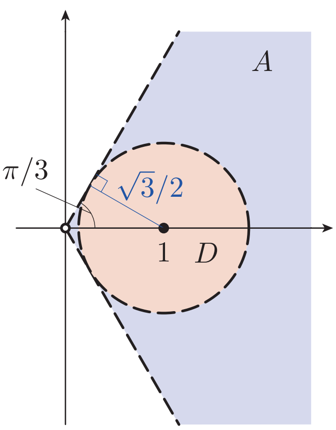


Figure 3.3 The disc D inside A

Exercise 3.2

Let $f(z) = 1 + z^4$.

(a) Prove that for each $w \neq 1$ the equation $f(z) = w$ has four distinct solutions.

(b) Determine an open disc D with centre $1 + i$ such that the restriction of f to D is one-to-one.

The functions $f(z) = z^3$ in Example 3.1 and $f(z) = 1 + z^4$ in Exercise 3.2 illustrate various possible types of local behaviour of an analytic function. Although these functions are not one-to-one (on their domains), they are one-to-one *near* certain points.

However, $f(z) = z^3$ is not one-to-one near 0. Indeed, a better description of the behaviour of $f(z) = z^3$ near 0 is *three-to-one*. To see why this is so, observe that if $w = z^3$, then $|z| < 1$ if and only if $|w| < 1$. Therefore f maps the punctured disc $\{z : 0 < |z| < 1\}$ onto the punctured disc $\{w : 0 < |w| < 1\}$, and, by Example 3.1(a), each point in $\{w : 0 < |w| < 1\}$ is the image of three points in $\{z : 0 < |z| < 1\}$. Similarly, the behaviour of $f(z) = 1 + z^4$ near 0 can be described as *four-to-one*. These examples suggest the following definition.

Definition

Let f be a function that is analytic on a region $\mathcal{R}$, and let $\alpha \in \mathcal{R}$.

Then f is ***n*-to-one near *α*** if there is a region $\mathcal{S}$ inside $\mathcal{R}$ with $\alpha \in \mathcal{S}$ such that for each point w in $f(\mathcal{S}) - \{f(\alpha)\}$ there are exactly n points z in $\mathcal{S} - \{\alpha\}$ that satisfy $f(z) = w$.

The definition is illustrated in Figure 3.4 for the case $n = 3$.

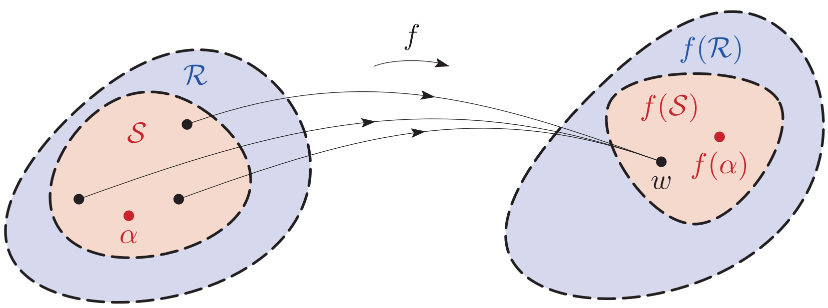


Figure 3.4 The function f is three-to-one near α

As we have seen, the function $f(z) = z^3$ is three-to-one near 0. Example 3.1(b) demonstrates that f is one-to-one near the point 1, since f is a one-to-one function when restricted to a suitably small open disc centred at 1.

The following key result shows that the behaviour of an analytic function f near a point α is determined by the Taylor series about α for f.

Theorem 3.2 Local Mapping Theorem

Let f be a function that is analytic on a region $\mathcal{R}$, and let $\alpha \in \mathcal{R}$. Suppose that the Taylor series about α for f has the form

$$f(z) = f(\alpha) + a_n(z-\alpha)^n + a_{n+1}(z-\alpha)^{n+1} + \cdots,$$

where $n \geq 1$ and $a_n \neq 0$. Then f is n-to-one near α.

This theorem is proved in Subsection 3.3.

By Taylor's Theorem (Theorem 3.1 of Unit B3), the kth Taylor coefficient of f at α is given by the formula $a_k = f^{(k)}(\alpha)/k!$. From this observation, we obtain the following corollary to the Local Mapping Theorem.

Corollary

Let f be a function that is analytic on a region $\mathcal{R}$, and let $\alpha \in \mathcal{R}$. Suppose that

$$f'(\alpha) = f''(\alpha) = \cdots = f^{(n-1)}(\alpha) = 0, \quad \text{but } f^{(n)}(\alpha) \neq 0,$$

where $n \geq 1$. Then f is n-to-one near α.

It is important to observe that the value of $f(\alpha)$ has no bearing on the type of local behaviour of f at α.

In the next example we apply the Local Mapping Theorem and its corollary to investigate the local behaviour of a given analytic function near several points of its domain.

Example 3.2

Describe the local behaviour of the function $f(z) = z^3 - z^2$ near each of the following points.

(a) $\alpha = 0$ (b) $\alpha = 1$ (c) $\alpha = \frac{2}{3}$

Solution

(a) Let $\alpha = 0$. Since the Taylor series about 0 for f is

$$f(z) = -z^2 + z^3,$$

the function f is two-to-one near 0, by the Local Mapping Theorem.

(b) Let $\alpha = 1$. Then

$$f'(\alpha) = 3\alpha^2 - 2\alpha = 1 \neq 0,$$

so f is one-to-one near 1, by the corollary to the Local Mapping Theorem.

(c) Let $\alpha = \frac{2}{3}$. Then

$$f'(\alpha) = 3\alpha^2 - 2\alpha = 0,$$
$$f''(\alpha) = 6\alpha - 2 = 2 \neq 0,$$

so f is two-to-one near $\frac{2}{3}$, by the corollary to the Local Mapping Theorem.

Exercise 3.3

Let $f(z) = z - z^2$.

(a) Describe the local behaviour of the function f near each of the points $\alpha = 0$ and $\alpha = \frac{1}{2}$.

(b) Prove that the restriction of the function f to $\{z : |z| \leq r\}$ is one-to-one when $r = \frac{1}{2}$, and that it is not one-to-one when $r > \frac{1}{2}$.

(*Hint*: Show that if $z_1, z_2 \in \left\{z : |z| \leq \frac{1}{2}\right\}$ and $f(z_1) = f(z_2)$, then $z_1 = z_2$.)

3.2 Inverse functions

We are now going to use the Local Mapping Theorem to obtain an improved version of the Inverse Function Rule, first discussed in Subsection 3.2 of Unit A4. In that unit we showed that if $f\colon A \longrightarrow B$ is a one-to-one function whose inverse function f^{-1} is continuous at $\beta \in B$, and f is differentiable at $f^{-1}(\beta)$ with $f'\big(f^{-1}(\beta)\big) \neq 0$, then f^{-1} is differentiable at β with

$$\big(f^{-1}\big)'(\beta) = \frac{1}{f'(f^{-1}(\beta))}.$$

We can now give a much stronger version of this result for analytic functions.

Theorem 3.3 Inverse Function Rule

Let f be a one-to-one analytic function whose domain is a region $\mathcal{R}$. Then f^{-1} is analytic on $f(\mathcal{R})$ and

$$\big(f^{-1}\big)'(\beta) = \frac{1}{f'(f^{-1}(\beta))}, \quad \text{for } \beta \in f(\mathcal{R}).$$

Proof To prove the result we need to show that

1. $f'(\alpha) \neq 0$, for $\alpha \in \mathcal{R}$
2. f^{-1} is continuous on $f(\mathcal{R})$.

Then we can apply the version of the Inverse Function Rule from Unit A4 to deduce that f^{-1} is differentiable at each $\beta \in f(\mathcal{R})$ and that $\big(f^{-1}\big)'(\beta) = 1/f'\big(f^{-1}(\beta)\big)$.

To prove 1, note that if $f'(\alpha) = 0$ for some $\alpha \in \mathcal{R}$, then, by the Local Mapping Theorem, f is n-to-one near α, for some $n > 1$; this contradicts the hypothesis that f is one-to-one on $\mathcal{R}$.

To prove 2, let $\beta \in f(\mathcal{R})$ and put $\alpha = f^{-1}(\beta)$. We want to show that for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$|w - \beta| < \delta \implies |f^{-1}(w) - \alpha| < \varepsilon, \tag{3.1}$$

as illustrated in Figure 3.5. (This is the ε-δ definition of continuity, applied to f^{-1} at β.)

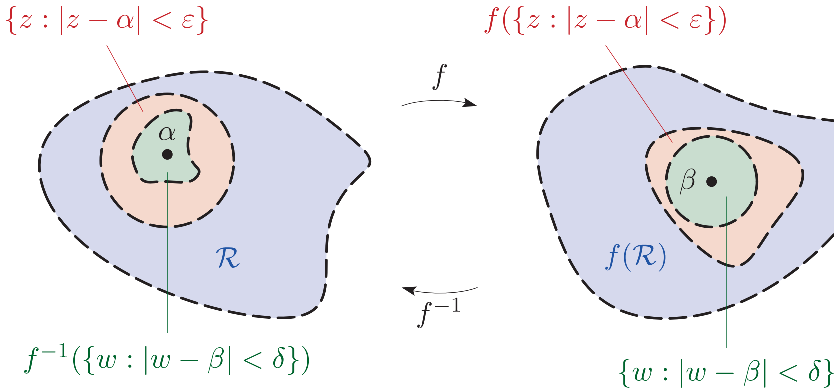


Figure 3.5 Functions f and f^{-1} mapping between $\mathcal{R}$ and $f(\mathcal{R})$

Now, given $\varepsilon > 0$, we know that $f(\{z : |z - \alpha| < \varepsilon\})$ is an open set, by the Open Mapping Theorem, so there exists $\delta > 0$ such that

$$\{w : |w - \beta| < \delta\} \subseteq f(\{z : |z - \alpha| < \varepsilon\})$$

(see the right-hand side of Figure 3.5), which implies that

$$f^{-1}(\{w : |w - \beta| < \delta\}) \subseteq \{z : |z - \alpha| < \varepsilon\}.$$

Hence implication (3.1) holds, as required. ■

In Subsection 4.1 of Unit B3 we introduced the inverse tan function, $\tan^{-1}$, and the inverse sin function, $\sin^{-1}$, and stated some of their properties. In particular, the function $\tan^{-1}$ is the inverse function of the restriction of tan to the strip $\{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$, and $\sin^{-1}$ is the inverse function of the restriction of sin to this strip. The domains of $\tan^{-1}$ and $\sin^{-1}$ are shown in Figure 3.6. (We justify the domain for $\tan^{-1}$ in Unit C3.)

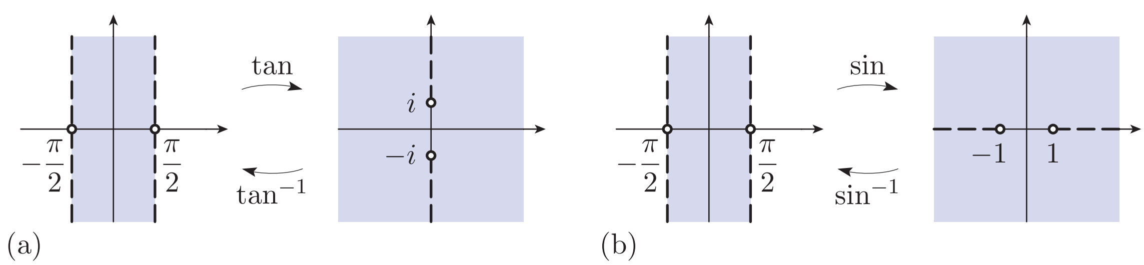


Figure 3.6 Domains and codomains of tan and sin and their inverse functions

Theorem 3.3 can be used to show that these inverse functions are analytic.

Example 3.3

Prove that the function

$$f(z) = \tan z \quad (-\pi/2 < \operatorname{Re} z < \pi/2)$$

has an inverse function f^{-1} which is analytic.

Solution

To show that f is one-to-one on the region $\mathcal{R} = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$, note that if $z_1, z_2 \in \mathcal{R}$, then

$$\begin{aligned} \tan z_1 = \tan z_2 &\implies \frac{\sin z_1}{\cos z_1} = \frac{\sin z_2}{\cos z_2} \\ &\implies \sin z_1 \cos z_2 = \sin z_2 \cos z_1 \\ &\implies \sin(z_1 - z_2) = 0 \\ &\implies z_1 - z_2 = n\pi, \quad \text{where } n \in \mathbb{Z}, \\ &\implies z_1 = z_2, \end{aligned}$$

because $|\operatorname{Re}(z_1 - z_2)| < \pi$. Since f is analytic on $\mathcal{R}$, we deduce, by the Inverse Function Rule, that f^{-1} is analytic on $f(\mathcal{R})$.

Furthermore, for $\beta \in f(\mathcal{R})$,

$$\left(f^{-1}\right)'(\beta) = (\tan^{-1})'(\beta) = \frac{1}{1+\beta^2},$$

as we saw in Subsection 4.1 of Unit B3.

Exercise 3.4

Prove that each of the following functions has an analytic inverse function.

(a) $f(z) = \sin z \quad (-\pi/2 < \operatorname{Re} z < \pi/2)$

(b) $f(z) = z - z^2 \quad \left(|z| < \frac{1}{2}\right)$

(*Hint*: For part (b), see Exercise 3.3.)

Exercise 3.5

Let f be a function that is analytic at a point α, and suppose that $f'(\alpha) \neq 0$. Prove that there is a region $\mathcal{S}$, with $\alpha \in \mathcal{S}$, such that the restriction of f to $\mathcal{S}$ has an analytic inverse function.

The Inverse Function Rule guarantees the existence of an analytic inverse function f^{-1} for any one-to-one analytic function f. Then, given a point $\beta = f(\alpha)$, it is natural to try to determine the Taylor series about β for f^{-1} in terms of the Taylor series about α for f. The next example illustrates one method for doing this, by determining the Taylor series for $h(w) = \sin^{-1} w$. (We found this Taylor series in Exercise 4.6 of Unit B3, by using the Integration Rule for power series.)

Example 3.4

Use the Taylor series about 0 for the function

$$f(z) = \sin z \quad (-\pi/2 < \operatorname{Re} z < \pi/2)$$

to determine the first three non-zero terms of the Taylor series about 0 for the function $f^{-1}(w) = \sin^{-1} w$.

Solution

First note that

$$f^{-1}(f(z)) = \sin^{-1}(\sin z) = z, \quad \text{for } -\pi/2 < \operatorname{Re} z < \pi/2. \tag{3.2}$$

Next observe that $f(0) = 0$, so we can apply the Composition Rule for power series (Theorem 4.3 of Unit B3) to calculate the Taylor series about 0 for $f^{-1} \circ f$ in terms of the Taylor series about 0 for f^{-1} and f. We know that

$$f(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad \text{for } |z| < \pi/2.$$

Since f is an odd function, it follows from Exercise 3.4 of Unit B3 that f^{-1} is an odd function too. Therefore the Taylor series about 0 for f^{-1} takes the form

$$f^{-1}(w) = b_1 w + b_3 w^3 + b_5 w^5 + \cdots$$

(no even terms) by Theorem 3.2(b) of Unit B3. Using equation (3.2) and the Composition Rule for power series, we see that

$$\begin{aligned} z &= b_1\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) + b_3\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^3 \\ &\quad + b_5\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^5 + \cdots \\ &= b_1\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) + b_3\left(z^3 + 3z^2\left(-\frac{z^3}{3!}\right) + \cdots\right) \\ &\quad + b_5\left(z^5 + \cdots\right) + \cdots. \end{aligned}$$

Equating the coefficients of $z, z^3, z^5, \ldots$, we obtain a sequence of equations for the coefficients $b_1, b_3, b_5, \ldots$:

$$\begin{aligned} &z: && 1 = b_1 && \Longrightarrow b_1 = 1, \\ &z^3: && 0 = -\frac{1}{3!}b_1 + b_3 && \Longrightarrow b_3 = \frac{1}{6}b_1 = \frac{1}{6}, \\ &z^5: && 0 = \frac{1}{5!}b_1 - \frac{3}{3!}b_3 + b_5 && \Longrightarrow b_5 = \frac{1}{2}b_3 - \frac{1}{5!}b_1 = \frac{3}{40}. \end{aligned}$$

Hence the required Taylor series is

$$\sin^{-1} w = w + \frac{1}{6}w^3 + \frac{3}{40}w^5 + \cdots.$$

Example 3.4 was fairly straightforward because the Taylor series for f and f^{-1} were both about 0. In general, given the Taylor series about a point α for f,

$$f(z) = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots,$$

we wish to find the Taylor series about the point $\beta = f(\alpha)$ for f^{-1},

$$f^{-1}(w) = b_0 + b_1(w-\beta) + b_2(w-\beta)^2 + \cdots.$$

Since $\beta = f(\alpha) = a_0$ and $\alpha = f^{-1}(\beta) = b_0$, these two Taylor series can be written in the form

$$f(z) - \beta = a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots$$

and

$$f^{-1}(w) - \alpha = b_1(w-\beta) + b_2(w-\beta)^2 + \cdots.$$

Hence the identity

$$\begin{aligned} z - \alpha &= f^{-1}(f(z)) - \alpha \\ &= b_1\big(a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots\big) \\ &\quad + b_2\big(a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots\big)^2 + \cdots \end{aligned}$$

can be used to obtain $b_1, b_2, \ldots$ in terms of $a_1, a_2, \ldots$. The method, known informally as 'inverting a Taylor series', is summarised in the following strategy.

Strategy for inverting a Taylor series

Given the Taylor series about α for f,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n,$$

where $a_1 = f'(\alpha) \neq 0$, we can find the Taylor series about $\beta = f(\alpha)$ for f^{-1},

$$f^{-1}(w) = \sum_{n=0}^{\infty} b_n(w-\beta)^n,$$

by putting $b_0 = \alpha$ and equating the powers of $(z-\alpha)$ in the identity

$$\begin{aligned} z - \alpha &= b_1\big(a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots\big) \\ &\quad + b_2\big(a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots\big)^2 + \cdots \end{aligned}$$

to obtain equations for $b_1, b_2, \ldots$ in terms of $a_1, a_2, \ldots$.

Recall that these Taylor series representations about α and β are valid on suitable open discs with centres α and β, respectively.

Note that the assumption that $f'(\alpha) \neq 0$ in the strategy is needed in order to guarantee that f is one-to-one near α (so that the restriction of f to a region $\mathcal{S}$ about α has an analytic inverse function).

Exercise 3.6

Use the strategy for inverting a Taylor series to invert the Taylor series about 0 for each of the following functions. Give the first three non-vanishing terms in each case.

(a) $f(z) = e^z$

(b) $f(z) = z - z^2$

3.3 Proofs of the mapping theorems

This subsection contains some challenging, but illuminating, proofs, which may be omitted on a first reading. We begin by proving the Open Mapping Theorem.

Theorem 3.1 Open Mapping Theorem

Let f be a function that is analytic and non-constant on a region $\mathcal{R}$, and let G be an open subset of $\mathcal{R}$. Then $f(G)$ is open.

Proof To prove that $f(G)$ is open, we need to show that if $\beta \in f(G)$, then there exists $\varepsilon > 0$ such that

$$\{w : |w - \beta| < \varepsilon\} \subseteq f(G).$$

Since $\beta \in f(G)$, we know that there exists $\alpha \in G$ such that $f(\alpha) = \beta$. Furthermore, the solutions of the equation $f(z) = \beta$ are isolated (by Theorems 5.2 and 5.3 of Unit B3, since $z \longmapsto f(z) - \beta$ is analytic and non-constant). Therefore we can choose an open disc D in G with centre α and radius sufficiently small so that $f(z) \neq \beta$, for $z \in D - \{\alpha\}$. Thus if C is a circle in D with centre α, then the image $f(C)$ is a closed contour that does not pass through β (see Figure 3.7).

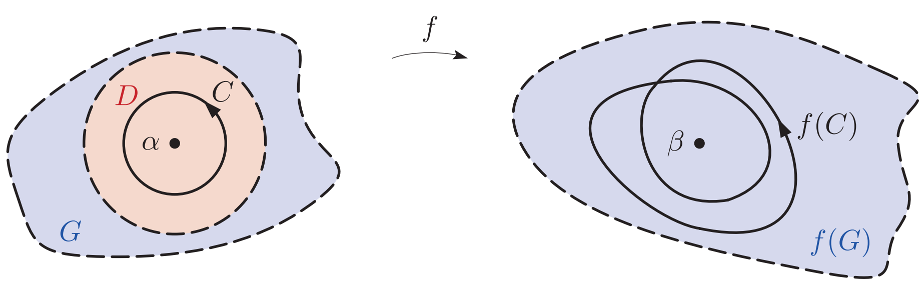


Figure 3.7 Image of a circle C under the function f

The set $f(C)$ is compact, because it is the continuous image of a compact set (Theorem 5.4 of Unit A3), so the complement of $f(C)$ is open. Hence we can choose $\varepsilon > 0$ so that $\{w : |w - \beta| < \varepsilon\}$ lies in the complement of $f(C)$ (see Figure 3.8).

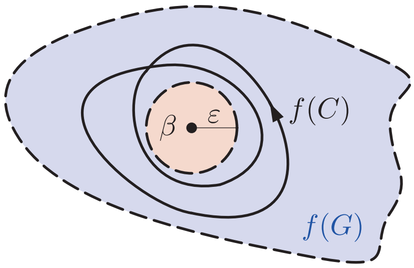


Figure 3.8 A disc $\{w : |w - \beta| < \varepsilon\}$ in the complement of $f(C)$

By Theorem 1.3, the winding number of $f(C)$ around each point of the disc $\{w : |w - \beta| < \varepsilon\}$ is equal to $\text{Wnd}(f(C), \beta)$. Now,

$$\text{Wnd}(f(C), \beta) \geq 1, \tag{3.3}$$

by the corollary to the Argument Principle (Theorem 2.3), because the equation $f(z) = \beta$ has at least one solution (namely α) inside C. Hence

$$\text{Wnd}(f(C), w) \geq 1, \quad \text{for } |w - \beta| < \varepsilon.$$

Thus, by the corollary to the Argument Principle again, the equation $f(z) = w$ has at least one solution z inside C, for each w such that $|w - \beta| < \varepsilon$. Hence $\{w : |w - \beta| < \varepsilon\} \subseteq f(G)$, as required. ■

Next we prove the Local Mapping Theorem.

Theorem 3.2 Local Mapping Theorem

Let f be a function that is analytic on a region $\mathcal{R}$, and let $\alpha \in \mathcal{R}$. Suppose that the Taylor series about α for f has the form

$$f(z) = f(\alpha) + a_n(z - \alpha)^n + a_{n+1}(z - \alpha)^{n+1} + \cdots,$$

where $n \geq 1$ and $a_n \neq 0$. Then f is n-to-one near α.

Proof To begin, we observe that

$$\begin{aligned} f(z) - f(\alpha) &= a_n(z - \alpha)^n + a_{n+1}(z - \alpha)^{n+1} + \cdots \\ &= (z - \alpha)^n(a_n + a_{n+1}(z - \alpha) + \cdots) \\ &= (z - \alpha)^n g(z), \end{aligned} \tag{3.4}$$

where the function $g(z) = a_n + a_{n+1}(z - \alpha) + \cdots$ is analytic at α with $g(\alpha) = a_n \neq 0$. Thus the function $h(z) = f(z) - f(\alpha)$ has a zero of order n at α. Equation (3.4) shows that the theorem is at least believable since, if z is near α, then

$$f(z) - f(\alpha) \approx (z - \alpha)^n g(\alpha),$$

and the function $z \longmapsto (z - \alpha)^n g(\alpha)$ is n-to-one near α.

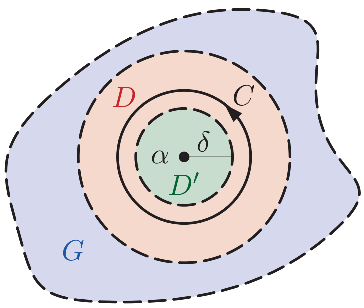


Figure 3.9 Disc $D' = \{z : |z-\alpha| < \delta\}$ inside C

Now assume that $n = 1$, so $f'(\alpha) = a_1 \neq 0$. Let $\beta = f(\alpha)$ and choose the open disc D, the circle C and radius $\varepsilon > 0$ as in the proof of Theorem 3.1. Choose D small enough that $g(z) \neq 0$, for $z \in D$, in which case $f(z) - f(\alpha) = (z-\alpha)g(z)$ has a simple zero at α and no other zeros in D. Inequality (3.3) now takes the more precise form

$$\operatorname{Wnd}(f(C), \beta) = 1,$$

by the corollary to the Argument Principle. Hence, by Theorem 1.3,

$$\operatorname{Wnd}(f(C), w) = 1, \quad \text{for } |w - \beta| < \varepsilon. \tag{3.5}$$

Because f is continuous at α, we can choose $\delta > 0$ so small that $D' = \{z : |z-\alpha| < \delta\}$ lies inside C (see Figure 3.9) and

$$|z - \alpha| < \delta \implies |f(z) - \beta| < \varepsilon.$$

Hence, by equation (3.5), the restriction of f to D' is one-to-one.

Next suppose that $n > 1$. Since $g(\alpha) = a_n \neq 0$, we can choose a generalised logarithm function, $\operatorname{Log}_\theta$ say, such that the function

$$h(z) = \exp\left(\frac{1}{n}\operatorname{Log}_\theta(g(z))\right)$$

is analytic on an open disc E with centre α, and we then have

$$(h(z))^n = \left(\exp\left(\frac{1}{n}\operatorname{Log}_\theta(g(z))\right)\right)^n = g(z), \quad \text{for } z \in E.$$

Therefore equation (3.4) gives

$$f(z) - f(\alpha) = (z-\alpha)^n h(z)^n = (\phi(z))^n, \quad \text{for } z \in E,$$

where ϕ is the analytic function

$$\phi(z) = (z-\alpha)h(z) \quad (z \in E).$$

Thus we can write

$$f(z) = f(\alpha) + (\phi(z))^n = p(\phi(z)), \quad \text{for } z \in E,$$

where p is the polynomial function

$$p(\zeta) = f(\alpha) + \zeta^n \quad (\zeta \in \mathbb{C}).$$

Now, $\phi(\alpha) = 0$ and $\phi'(\alpha) = h(\alpha) \neq 0$. Thus we can apply the $n = 1$ case to ϕ to obtain an open disc E' centred at α such that the restriction of ϕ to E' is one-to-one. Observe that $\phi(E')$ is a region (by the corollary to the Open Mapping Theorem) which contains 0. By the Inverse Function Rule (Theorem 3.3), the restriction of ϕ to E' has an inverse function $\phi^{-1} : \phi(E') \longrightarrow E'$, which is analytic on $\phi(E')$ (see Figure 3.10).

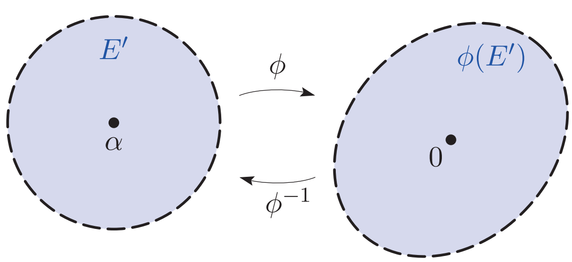


Figure 3.10 The function ϕ is a one-to-one mapping from E' to $\phi(E')$ with inverse function ϕ^{-1}

Next we choose an open disc Δ centred at 0 that is contained in $\phi(E')$, as shown in Figure 3.11, and define $\mathcal{S} = \phi^{-1}(\Delta)$. This set is a region, by the corollary to the Open Mapping Theorem, which contains $\alpha = \phi^{-1}(0)$. The restriction of ϕ to $\mathcal{S}$ is a one-to-one mapping from $\mathcal{S}$ onto the disc Δ.

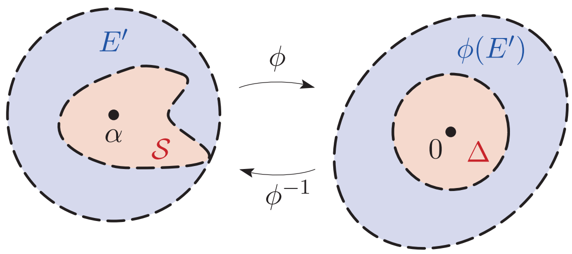


Figure 3.11 The function ϕ is a one-to-one mapping from $\mathcal{S}$ onto Δ

Let us now consider the restriction of the function f to $\mathcal{S}$. We observe that

$$f(\mathcal{S}) = p(\phi(\mathcal{S})) = p(\Delta),$$

as illustrated in Figure 3.12.

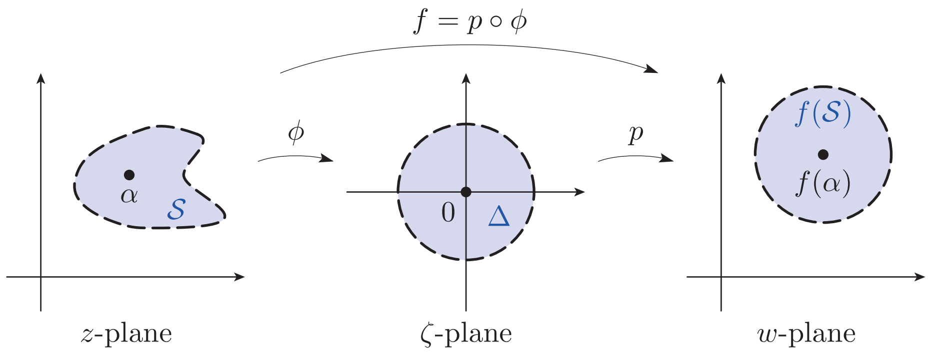


Figure 3.12 The function ϕ maps $\mathcal{S}$ to Δ, and the function p maps Δ to $f(\mathcal{S})$

Now we choose any point w in $f(\mathcal{S}) - \{f(\alpha)\}$. Since $p(\zeta) = f(\alpha) + \zeta^n$, we see that $p(\zeta) = w$ if and only if $\zeta^n = w - f(\alpha)$. This equation has exactly n complex solutions $\zeta_1, \zeta_2, \dots, \zeta_n$ (by Theorem 3.1 of Unit A1), which are equal in modulus, so they all lie in $\Delta - \{0\}$. Since $\phi\colon \mathcal{S} \longrightarrow \Delta$ is a one-to-one function, and $\phi(\alpha) = 0$, we see that for each integer $k = 1, 2, \dots, n$ there is a unique point z_k in $\mathcal{S} - \{\alpha\}$ with $\phi(z_k) = \zeta_k$. Therefore

$$f(z_k) = p(\phi(z_k)) = p(\zeta_k) = w, \quad \text{for } k = 1, 2, \dots, n.$$

Furthermore, if $z \in \mathcal{S}$ and $f(z) = w$, then $p(\phi(z)) = w$, so $\phi(z) = \zeta_k$, for some integer k, which implies that $z = z_k$.

This argument shows that there are *exactly* n points z in $\mathcal{S} - \{\alpha\}$ such that $f(z) = w$, so f is n-to-one near α. ■

Remark

Let f be the analytic function from the proof of the Local Mapping Theorem, with Taylor series

$$f(z) = f(\alpha) + a_n(z-\alpha)^n + a_{n+1}(z-\alpha)^{n+1} + \cdots$$

about α, where $a_n \neq 0$. The proof actually gives more precise information about the behaviour of f near α, in the following sense. Observe that the radius of $f(\mathcal{S})$ that is parallel to the real axis (shown in Figure 3.13) is the image under p of n evenly spaced radii in Δ. These are in turn the images of n smooth paths emerging from α in $\mathcal{S}$. By the Inverse Function Rule, ϕ^{-1} is conformal at 0 (it preserves angles), so these n smooth paths in $\mathcal{S}$ have evenly spaced tangent vectors at α (represented by arrows in Figure 3.13 for the case $n = 5$). Hence $\mathcal{S}$ is divided into n 'sectors', each of which is mapped onto $f(\mathcal{S})$ by f.

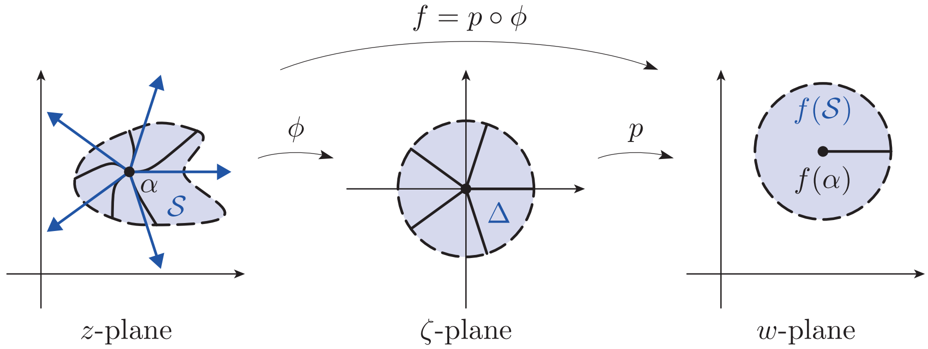


Figure 3.13 Five 'sectors' in $\mathcal{S}$, each mapped onto $f(\mathcal{S})$ by f

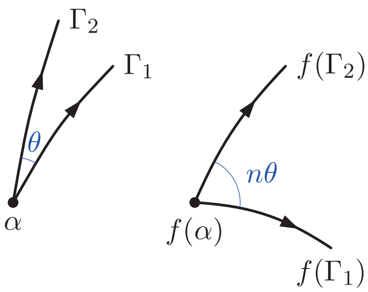


Figure 3.14 The angle θ is multiplied by a factor of n under f

Moreover, if Γ_1 and Γ_2 are *any* two smooth paths with initial point α such that the angle θ from Γ_1 to Γ_2 at α lies between $-\pi/n$ and π/n, then the angle from the path $f(\Gamma_1)$ to the path $f(\Gamma_2)$ at $f(\alpha)$ is $n\theta$ (see Figure 3.14).

This remark allows us to complete the proof of a result first stated in Unit A4.

Theorem 4.2 of Unit A4

Let f be a function that is analytic at a point α. Then f is conformal at α if and only if $f'(\alpha) \neq 0$.

Proof We saw in Unit A4 that if $f'(\alpha) \neq 0$, then f is conformal at α.

For the converse, suppose that $f'(\alpha) = 0$; we must prove that f is *not* conformal at α.

First, let us suppose that all the higher derivatives $f'(\alpha), f''(\alpha), f'''(\alpha), \ldots$ equal 0. Then f is constant on a disc centred at α, by Taylor's Theorem, so it certainly is not conformal at α.

Suppose next that

$$f'(\alpha) = f''(\alpha) = \cdots = f^{(n-1)}(\alpha) = 0, \quad \text{but } f^{(n)}(\alpha) \neq 0,$$

for some integer $n > 1$. As we have just seen, the angle from one smooth path with initial point α to another is multiplied by n under f (provided that the angle lies between $-\pi/n$ and π/n). But $n > 1$, so f does not preserve angles at α. That is, f is not conformal at α, as asserted. ■

Further exercises

Exercise 3.7

Prove that if f is a function that is analytic on the open unit disc $\{z : |z| < 1\}$, and

$$|f(z)| = \pi, \quad \text{for } |z| < 1,$$

then f is constant.

Exercise 3.8

Describe the local behaviour of each of the following functions near the given point α.

(a) $f(z) = \cos z, \quad \alpha = 0$

(b) $f(z) = e^z, \quad \alpha = 2\pi i$

(c) $f(z) = \sin z - z, \quad \alpha = 0$

Exercise 3.9

Invert the Taylor series for the following functions, giving the first three non-vanishing terms in each case.

(a) $f(z) = z^3 + 3z$ about $\alpha = 0$

(b) $f(z) = e^z$ about $\alpha = 1$

4 Extreme values of analytic functions

After working through this section, you should be able to:

- appreciate how the Open Mapping Theorem leads to the Local Maximum Principle
- understand the Maximum Principle and apply it to find the maximum modulus of an analytic function on a compact set, in appropriate cases
- understand and use the Minimum Principle, the Boundary Uniqueness Theorem and Schwarz's Lemma.

4.1 The Local Maximum Principle

In Subsection 3.1 we showed that non-constant analytic functions map regions to regions. This property has an application to the extrema (that is, the maxima and minima) of the modulus $|f|$ of an analytic function f. (The function $|f|$ is the function with the same domain as f and rule $z \longmapsto |f(z)|$.) We begin by defining the notion of a *local maximum*.

Definition

Let f be a function that is defined on a region $\mathcal{R}$. Then the function $|f|$ has a **local maximum** at a point $\alpha \in \mathcal{R}$ if there is some $r > 0$ such that $\{z : |z-\alpha| < r\} \subseteq \mathcal{R}$ and

$$|f(z)| \leq |f(\alpha)|, \quad \text{for } |z-\alpha| < r.$$

For example, if $f(z) = e^{-|z|^2}$, which is *not* an analytic function, then $|f| = f$ has a local maximum at 0, as illustrated by the graph of the surface $s = e^{-|z|^2}$ in Figure 4.1, in which the s-axis is a vertical axis perpendicular to the complex plane.

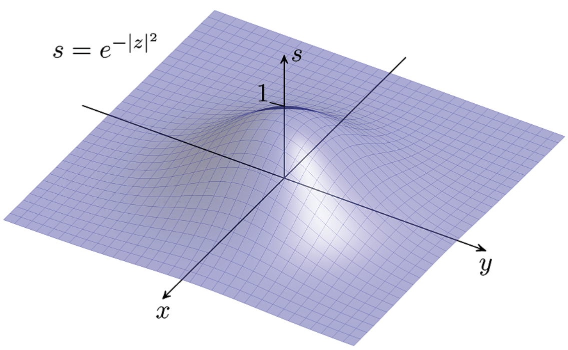


Figure 4.1 Graph of $s = e^{-|z|^2} = e^{-x^2-y^2}$

However, the graphs of the surfaces $s = |f(z)|$ for $f(z) = z^2$ and $f(z) = e^{-z}$, shown in Figure 4.2, suggest that the corresponding functions $|f|$ have no local maxima at all in $\mathbb{C}$.

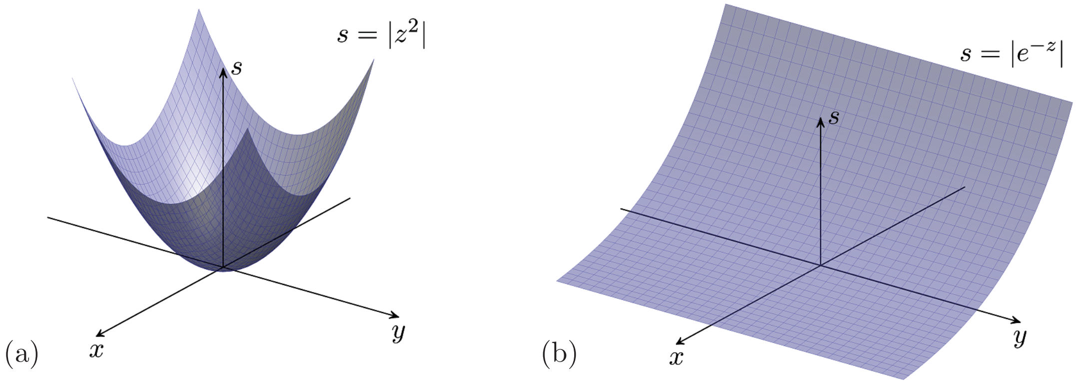


Figure 4.2 (a) Graph of $s = |z^2| = x^2 + y^2$ (b) Graph of $s = |e^{-z}| = e^{-x}$

The explanation for this absence of local maxima is as follows. Let f be a non-constant analytic function whose domain is a region $\mathcal{R}$, and let $\alpha \in \mathcal{R}$. Suppose, in order to reach a contradiction, that $|f|$ has a local maximum at α. Then there is some $r > 0$ such that $\{z : |z - \alpha| < r\} \subseteq \mathcal{R}$ and

$$|f(z)| \leq |f(\alpha)|, \quad \text{for } |z - \alpha| < r. \tag{4.1}$$

But if $D = \{z : |z - \alpha| < r\}$, then $f(D)$ is an open set containing $f(\alpha)$ (by the Open Mapping Theorem), so $f(D)$ contains an open disc, D' say, centred at $f(\alpha)$ (see Figure 4.3).

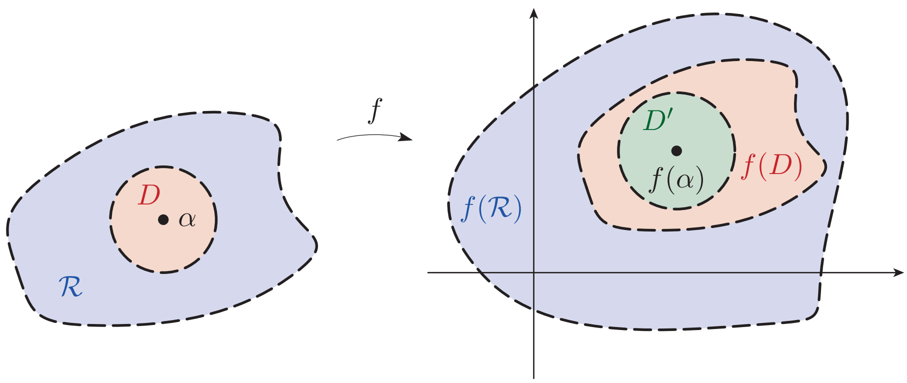


Figure 4.3 An open disc D' centred at $f(\alpha)$ inside $f(D)$

We now claim that there is some point w in D' such that

$$|w| > |f(\alpha)|.$$

If $f(\alpha) = 0$, then any point w in D' other than 0 itself satisfies this inequality. If $f(\alpha) \neq 0$, then such a w can be found by extending the line segment from 0 to $f(\alpha)$ slightly (see Figure 4.4).

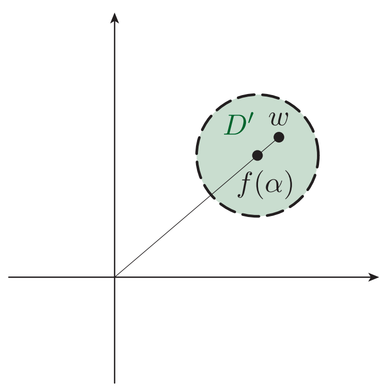


Figure 4.4 Point w in D' with $|w| > |f(\alpha)|$

Since $w \in D'$, and $D' \subseteq f(D)$, we have $w = f(z)$, for some $z \in D$. Hence

$$|f(z)| = |w| > |f(\alpha)|,$$

which contradicts inequality (4.1). Thus we have proved the following result.

Theorem 4.1 Local Maximum Principle

Let f be a function that is analytic and non-constant on a region $\mathcal{R}$. Then the function $|f|$ has no local maxima on $\mathcal{R}$.

Remark

This result shows yet again how different the behaviour of complex analytic functions is to that of standard real functions. For example, the complex function $f(z) = \cos z$ is non-constant and entire, so, by the Local Maximum Principle, the function $z \longmapsto |\cos z|$ has no local maxima on $\mathbb{C}$. But the real function $x \longmapsto |\cos x|$ has infinitely many local maxima, at $x = n\pi$, for $n \in \mathbb{Z}$ (see Figure 4.5, which illustrates these facts).

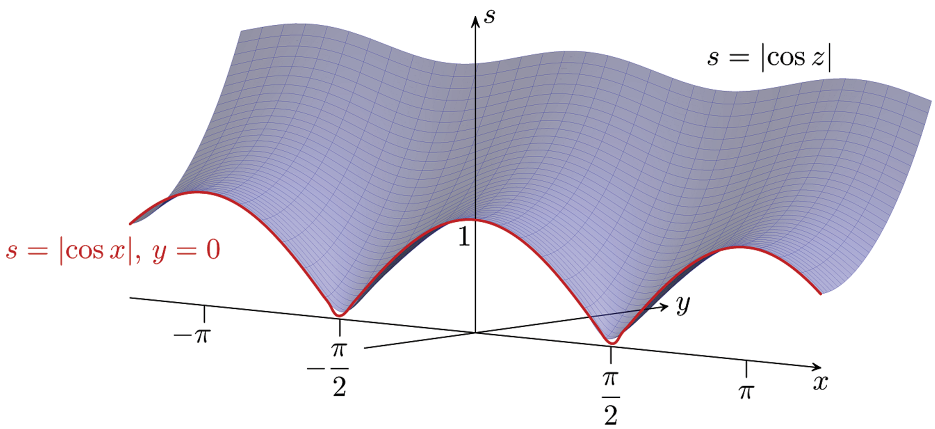


Figure 4.5 Surface $s = |\cos z|$, for $y \geq 0$, with the curve $s = |\cos x|$, $y = 0$, inscribed on it

The Local Maximum Principle often simplifies the process of finding the maximum modulus of an analytic function on a given set. Suppose, for example, that we want to determine

$$\max\{|z^2 - z - 1| : |z| \leq 1\}.$$

The function $f(z) = z^2 - z - 1$ is continuous on $\mathbb{C}$, and $\{z : |z| \leq 1\}$ is a compact set, so this maximum certainly exists by the Extreme Value Theorem (Theorem 5.2 of Unit A3). Since f is analytic and non-constant on the open disc $D = \{z : |z| < 1\}$, we deduce from the Local Maximum Principle that $|f|$ has no local maxima on D. Hence the maximum value of $|f|$ on $\{z : |z| \leq 1\}$ can be attained only on the boundary $\{z : |z| = 1\}$, that is,

$$\max\{|z^2 - z - 1| : |z| \leq 1\} = \max\{|z^2 - z - 1| : |z| = 1\}.$$

Thus we have reduced the problem of finding the maximum of $|f|$ on the set $\{z : |z| \leq 1\}$ to that of finding its maximum on $\{z : |z| = 1\}$, a considerable simplification (although still not easy!).

In the next subsection we formulate a general result of this type and then return to the problem above (in Example 4.1).

4.2 The Maximum Principle

First we recall from Subsection 5.1 of Unit A3 various notions about sets, which are illustrated in Figure 4.6.

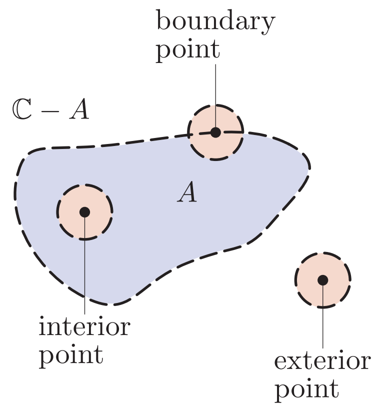


Figure 4.6 Interior point, exterior point and boundary point of a set A

Definitions

Let A be a subset of $\mathbb{C}$, and let $\alpha \in \mathbb{C}$. Then

- α is an **interior point** of A if there is an open disc centred at α that lies entirely in A
- α is an **exterior point** of A if there is an open disc centred at α that lies entirely outside A
- α is a **boundary point** of A if each open disc centred at α contains at least one point of A and at least one point of $\mathbb{C} - A$.

These three types of points form the **interior** of A, written $\operatorname{int} A$, the **exterior** of A, written $\operatorname{ext} A$, and the **boundary** of A, written ∂A.

It is evident that the sets $\operatorname{int} A$, ∂A and $\operatorname{ext} A$ are always disjoint and that

$$\operatorname{int} A \cup \partial A \cup \operatorname{ext} A = \mathbb{C}.$$

Thus

$$\operatorname{int} A \cup \partial A = \mathbb{C} - \operatorname{ext} A$$

is always closed (since $\operatorname{ext} A$ is open, by Theorem 5.5 of Unit A3), and it is called the *closure* of A.

Definition

The **closure** $\overline{A}$ of a set A in $\mathbb{C}$ is

$$\overline{A} = \operatorname{int} A \cup \partial A.$$

For our purposes, the notation $\overline{\mathcal{R}}$, where as usual $\mathcal{R}$ is a region, is a handy shorthand for the union of $\mathcal{R}$ with its boundary $\partial\mathcal{R}$. For example, if $\mathcal{R} = \{z : |z| < 1\}$, then $\partial\mathcal{R} = \{z : |z| = 1\}$, so $\overline{\mathcal{R}} = \{z : |z| \leq 1\}$.

Theorem 4.2 Maximum Principle

Let f be a function that is analytic and non-constant on a bounded region $\mathcal{R}$, and continuous on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial\mathcal{R}$ such that

$$|f(z)| \leq |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}},$$

with strict inequality for any $z \in \mathcal{R}$.

In some texts this result is called the Maximum Modulus Theorem.

Proof First we note that $\overline{\mathcal{R}}$ is closed. Also, $\mathcal{R}$ is bounded, so

$$\mathcal{R} \subseteq \{z : |z| \leq M\},$$

for some $M > 0$. Hence

$$\partial\mathcal{R} \subseteq \{z : |z| \leq M\},$$

since all points outside $\{z : |z| \leq M\}$ are exterior to $\mathcal{R}$. Hence

$$\overline{\mathcal{R}} \subseteq \{z : |z| \leq M\}.$$

Thus $\overline{\mathcal{R}}$ is a compact set. Since f is continuous on $\overline{\mathcal{R}}$, we see from the Extreme Value Theorem that there exists $\alpha \in \overline{\mathcal{R}}$ such that

$$|f(z)| \leq |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}}. \tag{4.2}$$

We are given that f is non-constant, so the function $|f|$ has no local maxima on $\mathcal{R}$, by the Local Maximum Theorem. It follows that $\alpha \in \partial\mathcal{R}$, and that inequality (4.2) is a strict inequality for any $z \in \mathcal{R}$. ■

Now we return to the example discussed at the end of Subsection 4.1.

Example 4.1

Show that

$$\max\{|z^2 - z - 1| : |z| \leq 1\} = \sqrt{5},$$

and determine where this maximum value is attained.

Solution

Since $f(z) = z^2 - z - 1$ is analytic and non-constant on the open disc $D = \{z : |z| < 1\}$ and continuous on $\overline{D} = \{z : |z| \leq 1\}$, it follows from the Maximum Principle that there exists a point α in $\partial D = \{z : |z| = 1\}$ such that

$$\max\{|f(z)| : |z| \leq 1\} = |f(\alpha)|.$$

Now, each point of ∂D can be expressed as e^{it}, for some $t \in [0, 2\pi)$, so we need to determine

$$\max\{|f(e^{it})| : 0 \leq t < 2\pi\}.$$

Observe that

$$f(e^{it}) = e^{2it} - e^{it} - 1 = e^{it}(e^{it} - 1 - e^{-it}) = e^{it}(2i\sin t - 1),$$

using the formula $\sin t = (e^{it} - e^{-it})/(2i)$. Therefore, since $|e^{it}| = 1$,

$$\left|f(e^{it})\right|^2 = |2i\sin t - 1|^2 = 1 + 4\sin^2 t.$$

Since $\sin^2 t \leq 1$, we see that

$$\left|f(e^{it})\right| \leq \sqrt{1+4} = \sqrt{5},$$

with equality if and only if $\sin t = \pm 1$. That is, equality is attained if and only if $t = \pi/2$ or $t = 3\pi/2$. Therefore the maximum is attained at the points $\pm i$ on ∂D, and at no other points on ∂D.

Thus we have shown that $\max\{|f(e^{it})| : 0 \le t < 2\pi\} = \sqrt{5}$, and hence

$$\max\{|z^2 - z - 1| : |z| \le 1\} = \sqrt{5}.$$

The Maximum Principle tells us that this maximum can be attained only on ∂D, so $\pm i$ are the only points in $\overline{D}$ at which the maximum is attained.

Exercise 4.1

Determine each of the following maxima, and find the points at which they are attained.

(a) $\max\{|z^2 + 1| : |z| \le 1\}$

(b) $\max\{|z^2 - 2| : |z - 1| \le 1\}$

(c) $\max\{|z^3 - 1| : 0 \le \operatorname{Re} z \le 1,\ 0 \le \operatorname{Im} z \le 1\}$

(*Hint*: In part (c) you will need to consider the four sides of the square separately.)

We conclude this section by stating a number of corollaries to the Maximum Principle.

Corollary Minimum Principle

Let f be a function that is analytic and non-constant on a bounded region $\mathcal{R}$, and continuous and non-zero on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial\mathcal{R}$ such that

$$|f(z)| \ge |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}},$$

with strict inequality for any $z \in \mathcal{R}$.

Proof Since f is non-zero on $\overline{\mathcal{R}}$, the function $g = 1/f$ is analytic on $\mathcal{R}$ and continuous on $\overline{\mathcal{R}}$. Also, g is non-constant because f is non-constant. Hence, by the Maximum Principle, there is a point α on $\partial\mathcal{R}$ such that

$$|g(z)| \le |g(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}}.$$

Since $|g(z)| = |1/f(z)| = 1/|f(z)|$, we deduce that

$$|f(z)| \ge |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}}. \tag{4.3}$$

Since $|f(z)| = |f(\alpha)|$ if and only if $|g(z)| = |g(\alpha)|$, we see from the Maximum Principle that inequality (4.3) is a strict inequality for any $z \in \mathcal{R}$, as required. ■

Remark

Notice the assumption here that f is non-zero on $\overline{\mathcal{R}}$. Without this, the result would be false. For example, if $f(z) = z$ and $\mathcal{R} = \{z : |z| < 1\}$, then $|f(z)| = 1$, for $z \in \partial\mathcal{R}$, but

$$\min\{|f(z)| : |z| \le 1\} = |f(0)| = 0.$$

In general, if $f(\alpha) = 0$ for some $\alpha \in \mathcal{R}$, then

$$\min\{|f(z)| : z \in \mathcal{R}\} = |f(\alpha)| = 0.$$

Exercise 4.2

Determine

$$\min\{\left|\exp(z^2)\right| : |z| \le 1\},$$

and find the points at which the minimum is attained.

The next corollary is related to Cauchy's Integral Formula (Theorem 2.1 of Unit B2), which says that

$$f(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\alpha}\,dz,$$

where f is a function that is analytic on a simply connected region containing the simple-closed contour Γ, and α is a point inside Γ. The formula tells us that the value of f at each point inside Γ is determined by the values of f on Γ. This last assertion holds in the following more general situation (where, unlike in Cauchy's Integral Formula, the region being considered need not be simply connected).

Corollary Boundary Uniqueness Theorem

Let f and g be functions that are analytic on a bounded region $\mathcal{R}$ and continuous on $\overline{\mathcal{R}}$. If $f = g$ on $\partial\mathcal{R}$, then $f = g$ on $\mathcal{R}$.

Exercise 4.3

Prove the Boundary Uniqueness Theorem by applying the Maximum Principle to the function $h = f - g$.

Our final corollary, which is known as Schwarz's Lemma (where Schwarz is pronounced 'shvarts') plays an important role in showing that certain analytic functions must take a particular form (see, for example, Exercise 4.4).

Corollary Schwarz's Lemma

Let f be a function that is analytic on the open disc $\{z : |z| < R\}$, with $f(0) = 0$, and suppose that

$$|f(z)| \leq M, \quad \text{for } |z| < R.$$

Then

$$|f(z)| \leq (M/R)|z|, \quad \text{for } |z| < R.$$

Remark

Schwarz's Lemma has the following simple geometric interpretation: if the analytic function f maps the open disc $\{z : |z| < R\}$ into the open disc $\{z : |z| < M\}$, with $f(0) = 0$, then the image of any point z on the circle Γ with equation $|z| = r$ lies inside or on the circle $|w| = (M/R)r$, as illustrated in Figure 4.7.

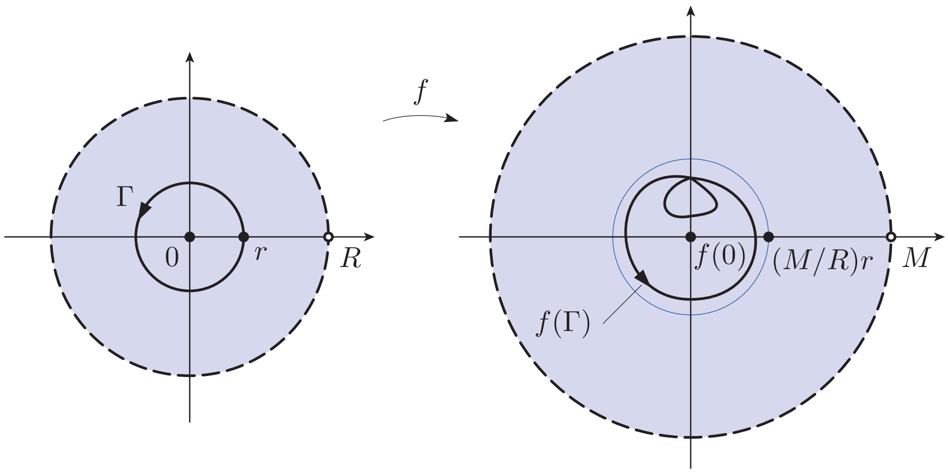


Figure 4.7 The image of the circle $\{z : |z| = r\}$ under f lies inside or on the circle $\{w : |w| = (M/R)r\}$

Proof Consider the Taylor series about 0 for f, which takes the form

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \quad \text{for } |z| < R,$$

since $f(0) = 0$. Then

$$\frac{f(z)}{z} = a_1 + a_2 z + a_3 z^2 + \cdots, \quad \text{for } 0 < |z| < R.$$

Thus the function

$$g(z) = a_1 + a_2 z + a_3 z^2 + \cdots \quad (|z| < R)$$

provides an analytic extension of $z \longmapsto f(z)/z$ to $\{z : |z| < R\}$.

We now apply the Maximum Principle to g on the open disc $\{z : |z| < r\}$, where $0 < r < R$ (we cannot take $r = R$ here, since f (and hence g) is not known to be continuous on $\{z : |z| \leq R\}$).

The Maximum Principle gives a point α with $|\alpha| = r$ such that

$$|g(z)| \leq |g(\alpha)| = \frac{|f(\alpha)|}{|\alpha|} \leq \frac{M}{r}, \quad \text{for } |z| \leq r.$$

(If g is constant, then the Maximum Principle does not apply. However, the inequality is still satisfied, with any choice of α such that $|\alpha| = r$.)

This inequality holds for all r such that $|z| \leq r < R$. Hence

$$|g(z)| \leq \frac{M}{R}, \quad \text{for } |z| < R.$$

It follows that

$$\left|\frac{f(z)}{z}\right| \leq \frac{M}{R}, \quad \text{for } 0 < |z| < R,$$

so

$$|f(z)| \leq \left(\frac{M}{R}\right)|z|, \quad \text{for } 0 < |z| < R.$$

Since this inequality evidently holds for $z = 0$ as well, the proof of Schwarz's Lemma is complete. ■

We end this section with an exercise that shows the power of Schwarz's Lemma.

Exercise 4.4

Suppose that f is a one-to-one analytic function with domain $D = \{z : |z| < 1\}$ such that

$$f(0) = 0 \quad \text{and} \quad f(D) = D.$$

Deduce that f is of the form $f(z) = \lambda z$, where $|\lambda| = 1$; that is, f is a rotation.

(*Hint*: Apply Schwarz's Lemma to both f and f^{-1} to show that $|f(z)| = |z|$, for $|z| < 1$, and then prove that the function $g(z) = f(z)/z$ is constant.)

Further exercises

Exercise 4.5

Determine each of the following maxima, and find the points at which they are attained.

(a) $\max\{|z^2+2| : |z| \leq 1\}$

(b) $\max\{|z^2-2| : |z-i| \leq 1\}$

(c) $\max\left\{\left|e^{z^2}\right| : -1 \leq \operatorname{Re} z \leq 1,\ -1 \leq \operatorname{Im} z \leq 1\right\}$

(d) $\max\{|\tan z| : -\pi/4 \leq \operatorname{Re} z \leq \pi/4,\ -1 \leq \operatorname{Im} z \leq 1\}$

(*Hint*: For part (d), use the formulas

$$|\tan z|^2 = \frac{|\sin z|^2}{|\cos z|^2} = \frac{\sin^2 x + \sinh^2 y}{\cos^2 x + \sinh^2 y},$$

which follow from the identities

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \quad \text{and} \quad |\cos z|^2 = \cos^2 x + \sinh^2 y$$

derived in Example 4.4 and Exercise 4.7 of Unit A2.)

Exercise 4.6

Under the assumptions of Schwarz's Lemma, show that

(a) $|f'(0)| \leq M/R$

(b) if $|f(z_0)| = (M/R)|z_0|$, for some $|z_0| < R$, or if $|f'(0)| = M/R$, then

$$f(z) = \lambda z \quad (|z| < R),$$

where λ is a constant with $|\lambda| = M/R$.

History of Schwarz's Lemma

Hermann Amandus Schwarz (1843–1921) was a German mathematician who made a wide range of advances in complex analysis and other branches of mathematics. His name was attached to Schwarz's Lemma by the Greek mathematician Constantin Carathéodory (1873–1950), who generalised Schwarz's original version of the lemma to its current form.

The lemma was extended still further by the Austrian mathematician Georg Alexander Pick (1859–1942) to give a theorem now known as the Schwarz–Pick Theorem, which is of deep significance in analysis and geometry.

Hermann Amandus Schwarz

5 Uniform convergence

After working through this section, you should be able to:

- understand the definition of *uniform convergence*
- use Weierstrass' M-test to prove that various series of functions are uniformly convergent
- use Weierstrass' Theorem to prove that various functions defined by series are analytic, and to find their derivatives
- define the *zeta function* ζ and understand the role of Weierstrass' Theorem in proving that ζ is analytic.

5.1 The zeta function

Taylor series and Laurent series are useful tools for exploring analytic functions. However, a limitation of these series is that they can be used to represent analytic functions only on an open disc or an open annulus, and for some important analytic functions, such sets are inappropriate. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots, \tag{5.1}$$

where

$$\frac{1}{n^z} = n^{-z} = e^{-z \log n},$$

which was considered in Example 1.5 of Unit B3, is absolutely convergent for $\operatorname{Re} z > 1$. Thus the sum function for the series in equation (5.1) can be used to define a function with domain the open half-plane $\{z : \operatorname{Re} z > 1\}$. This function is of great importance in number theory, a fact first recognised (for real z) by the Swiss mathematician Leonhard Euler, whom you met in Book A. For this reason it was studied extensively in the nineteenth century by Riemann, who called it the *zeta function*.

Definition

The **zeta function** is the function

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots \quad (\operatorname{Re} z > 1).$$

We have already calculated several values of the zeta function using the Residue Theorem. For example,

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90},$$

by Example 4.1 and Exercise 4.1 of Unit C1. The same technique can be used to determine $\zeta(6), \zeta(8), \ldots$, but most values of the function ζ can be found only approximately.

The definition of the zeta function can be expressed as

$$\zeta(z) = \lim_{n\to\infty} \zeta_n(z) \quad (\operatorname{Re} z > 1),$$

where (ζ_n) is the sequence of partial sum functions

$$\zeta_n(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots + \frac{1}{n^z} \quad (z \in \mathbb{C}), \tag{5.2}$$

for $n = 1, 2, \ldots$. The graphs of the restrictions of some of these partial sum functions to $\{x \in \mathbb{R} : x > 1\}$ are shown in Figure 5.1. The graph of the 'limit function' $x \longmapsto \zeta(x)$, for $x > 1$, shows that $\zeta(x)$ grows arbitrarily large as x approaches 1, which is why we need the restriction $\operatorname{Re} z > 1$ in the definition of ζ.

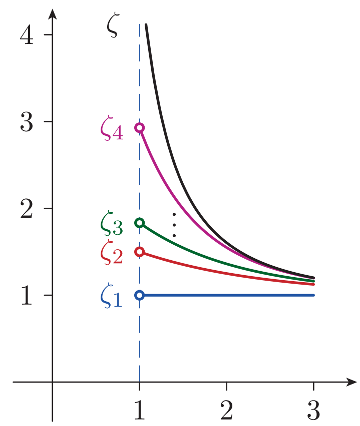


Figure 5.1 Graphs of the functions ζ_k and ζ

Each of the functions ζ_n is entire, since ζ_n is a sum of entire functions of the form $1/k^z = \exp(-z \log k)$, so it seems likely that the zeta function is analytic on its domain $\{z : \operatorname{Re} z > 1\}$. However, the only result that we have proved so far about a function defined by a series being analytic is the Differentiation Rule for power series (Theorem 2.3 of Unit B3). Since the series $1 + 1/2^z + 1/3^z + \cdots$ is not a power series, this theorem cannot be applied. In the next subsection we develop a type of convergence that will enable us to prove that the zeta function is indeed analytic.

5.2 Convergence of sequences of functions

There are various ways in which a sequence of functions (f_n) can converge to a limit function f; the following one is perhaps the simplest.

Definition

A sequence of functions (f_n) **converges pointwise** (to a **limit function** f) on a set E if, for each $z \in E$,

$$\lim_{n\to\infty} f_n(z) = f(z).$$

For example, the sequence of functions (ζ_n) defined by equation (5.2) converges pointwise to the zeta function ζ on the set $E = \{z : \operatorname{Re} z > 1\}$. A simpler example is the sequence of functions

$$f_n(z) = z^n, \quad n = 1, 2, \ldots.$$

Since $z^n \to 0$ as $n \to \infty$, for $|z| < 1$ (by Theorem 1.2(b) of Unit A3), we deduce that the sequence (f_n) converges pointwise to the zero function $f(z) = 0$ on the set $\{z : |z| < 1\}$. This behaviour is illustrated for *real* values of x in the interval $[0, 1]$ in Figure 5.2. Notice that $f_n(x) \to 0$ as $n \to \infty$, for $0 \le x < 1$, whereas $f_n(1) \to 1$ as $n \to \infty$.

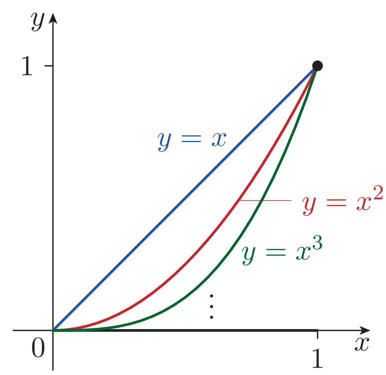


Figure 5.2 Graphs of $y = x^k$, for $k = 1, 2, 3$

Though useful, pointwise convergence is unfortunately not strong enough to give the type of convergence theorem that we require. Instead, we introduce a type of convergence that guarantees that $f_n(z) \to f(z)$ as $n \to \infty$ at a *uniform rate* as z varies over a set.

Definitions

A sequence of functions (f_n) **converges uniformly** (to a **limit function** f) on a set E if, for each $\varepsilon > 0$, there is an integer N such that

$$|f_n(z) - f(z)| < \varepsilon, \quad \text{for all } n > N \text{ and all } z \in E.$$

We also say that (f_n) is **uniformly convergent** on E, with limit function f.

This definition is illustrated in Figure 5.3 for two arbitrary points z_1 and z_2 in E and $n = N + 1$. The sequences

$$f_{N+1}(z_1), f_{N+2}(z_1), \ldots \quad \text{and} \quad f_{N+1}(z_2), f_{N+2}(z_2), \ldots$$

lie in the discs of radius ε with centres $f(z_1)$ and $f(z_2)$, respectively, and converge to those values.

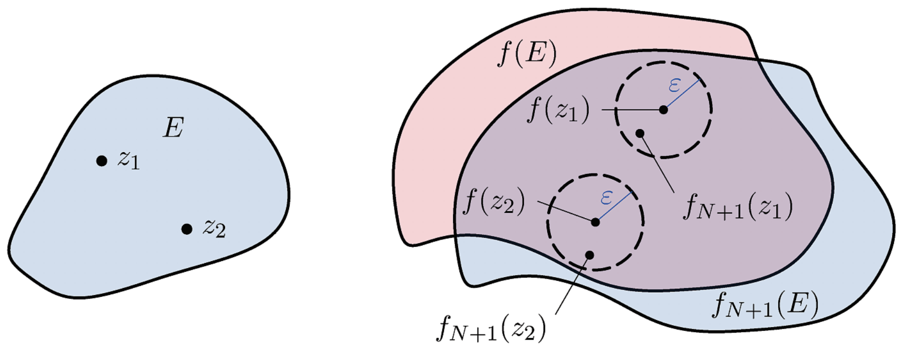


Figure 5.3 Points z_1 and z_2 in E, and image points $f(z_1)$, $f(z_2)$, $f_{N+1}(z_1)$ and $f_{N+1}(z_2)$

Remarks

1. To appreciate how the definition of uniform convergence differs from that of pointwise convergence, it helps to recast the definition of pointwise convergence in the following equivalent form:

   for each $\varepsilon > 0$, there is an integer N such that

   $$|f_n(z) - f(z)| < \varepsilon, \quad \text{for all } n > N.$$

   The choice of N here depends on ε *and* on z, whereas in the definition of uniform convergence the choice of N depends only on ε; that is, the same N works for all $z \in E$, hence the *uniformity* of the convergence.

2. It is clear that if (f_n) converges uniformly on E, then it converges uniformly on any subset of E. Also, if (f_n) converges uniformly to f on E, then (f_n) converges pointwise to f on E.

Example 5.1

Prove that the sequence $f_n(z) = z^n$, $n = 1, 2, \ldots$, is uniformly convergent on $E = \left\{z : |z| \le \frac{1}{2}\right\}$.

Solution

We know that if (f_n) converges *uniformly* to a limit function, then it also converges *pointwise* to the same limit function. With this fact in mind, we begin by showing that (f_n) converges pointwise to some function f, and then we use this function f to prove that (f_n) converges uniformly to f.

For each $z \in E$, we have $|z| \le \frac{1}{2}$, so

$$f_n(z) = z^n \to 0 \text{ as } n \to \infty.$$

Hence (f_n) converges pointwise to the function $f(z) = 0$ on E.

Next,

$$|f_n(z) - f(z)| = |z|^n \le \left(\tfrac{1}{2}\right)^n, \quad \text{for } n = 1, 2, \ldots \text{ and all } z \in E.$$

Therefore to prove uniform convergence we need only show that for each $\varepsilon > 0$ there is an integer N such that

$$\left(\tfrac{1}{2}\right)^n < \varepsilon, \quad \text{for all } n > N.$$

Since $\left(\left(\frac{1}{2}\right)^n\right)$ is a basic null sequence, this is clearly possible.

Hence (f_n) converges uniformly to the function $f(z) = 0$ on E.

The solution to Example 5.1 illustrates one way to prove that a sequence of functions is uniformly convergent, which we summarise in the following strategy.

Strategy for proving uniform convergence

To prove that a sequence of functions (f_n) is uniformly convergent on a set E, proceed as follows.

1. Determine the limit function f by evaluating

$$f(z) = \lim_{n\to\infty} f_n(z), \quad \text{for } z \in E.$$

2. Find a null sequence (a_n) of positive terms such that

$$|f_n(z) - f(z)| \le a_n, \quad \text{for } n = 1, 2, \ldots \text{ and all } z \in E.$$

Using this strategy we can show, as in the solution to Example 5.1, that the sequence of functions $f_n(z) = z^n$, $n = 1, 2, \ldots$, converges uniformly to the function $f(z) = 0$ on *any* closed disc of the form $E = \{z : |z| \le r\}$, where $0 < r < 1$. In this case the null sequence in step 2 of the strategy is (r^n).

Notice, however, that this sequence of functions does not converge uniformly to its limit function $f(z) = 0$ on the set $E = \{z : |z| < 1\}$. Roughly speaking, this is because $z^n \to 0$ as $n \to \infty$ more and more slowly as z approaches the boundary of $\{z : |z| < 1\}$ (see Figure 5.2). More precisely, if $\varepsilon = \frac{1}{2}$ say, then there is no positive integer N such that

$$|f_n(z) - f(z)| = |z|^n < \tfrac{1}{2}, \quad \text{for all } n > N \text{ and all } z \in E,$$

because each complex number $z \in E$ that satisfies $|z| \geq \left(\frac{1}{2}\right)^{1/n}$ is such that

$$|z|^n \geq \tfrac{1}{2}.$$

Exercise 5.1

Prove that the sequence

$$f_n(z) = \frac{1}{1+z^n}, \quad n = 1, 2, \ldots,$$

is uniformly convergent on $E = \{z : |z| \leq r\}$, where $0 < r < 1$.

To fulfil our objective of proving that the zeta function is analytic, we now adapt the concepts of pointwise and uniform convergence to series of functions. The various conventions associated with series of complex numbers are readily adapted to series of complex functions. For instance, if necessary, such a series can start with a value of n other than 1.

The next definition extends the concept of the sum function of a power series on a set given in Subsection 2.1 of Unit B3.

Definitions

If (ϕ_n) is a sequence of functions, then the series of functions

$$\sum_{n=1}^{\infty} \phi_n = \phi_1 + \phi_2 + \cdots$$

converges pointwise on a set E if the sequence of **partial sum functions** (f_n), where

$$f_n(z) = \phi_1(z) + \phi_2(z) + \cdots + \phi_n(z), \quad n = 1, 2, \ldots,$$

converges pointwise on E.

The series of functions **converges uniformly** on a set E, or is **uniformly convergent** on E, if the sequence of partial sum functions converges uniformly on E.

The limit function f of the sequence (f_n) is called the **sum function** of $\sum_{n=1}^{\infty} \phi_n$ on E, written

$$f(z) = \sum_{n=1}^{\infty} \phi_n(z) \quad (z \in E).$$

For example, if

$$\phi_n(z) = z^n, \quad n = 0, 1, 2, \ldots,$$

then the partial sum functions of

$$\sum_{n=0}^{\infty} \phi_n$$

are

$$\begin{aligned} f_n(z) &= \phi_0(z) + \phi_1(z) + \cdots + \phi_n(z) \\ &= 1 + z + \cdots + z^n \\ &= \frac{1 - z^{n+1}}{1 - z}, \quad \text{for } z \neq 1. \end{aligned}$$

Hence the sum function of $\sum_{n=0}^{\infty} \phi_n$ on $\{z : |z| < 1\}$ is

$$f(z) = \lim_{n\to\infty} f_n(z) = \frac{1}{1-z} \quad (|z| < 1).$$

In view of Example 5.1 and the discussion following it, we might expect the convergence of this series to be uniform on each set $\{z : |z| \leq r\}$, where $0 < r < 1$. Rather than prove this directly, we can use the following test for uniform convergence of series, which is widely applicable. It is named after the German mathematician Karl Weierstrass, whom you met in Unit B4.

Theorem 5.1 Weierstrass' M-test

Let (ϕ_n) be a sequence of functions defined on a set E, and suppose that there is a sequence of positive numbers (M_n) such that

1. $|\phi_n(z)| \leq M_n$, for $n = 1, 2, \ldots$ and all $z \in E$
2. $\sum_{n=1}^{\infty} M_n$ is convergent.

Then the series $\sum_{n=1}^{\infty} \phi_n$ is uniformly convergent on E.

We will refer to this result as the *M-test*, since we introduce a different result called Weierstrass' Theorem in the next subsection.

When applying the M-test, make sure that the sequence (M_n) that you choose satisfies *both* assumptions 1 and 2. You may need to refer to results from Section 1 of Unit B3 to prove that $\sum_{n=1}^{\infty} M_n$ converges.

Proof First note that, by assumptions 1 and 2, $\sum_{n=1}^{\infty} |\phi_n(z)|$ is convergent for each $z \in E$, by the Comparison Test (Theorem 1.6 of Unit B3). Hence $\sum_{n=1}^{\infty} \phi_n(z)$ is convergent for each $z \in E$, by the Absolute Convergence Test (Theorem 1.7 of Unit B3), so the sum function

$$f(z) = \sum_{n=1}^{\infty} \phi_n(z)$$

exists for each $z \in E$. As usual, we denote the partial sum functions by

$$f_n(z) = \phi_1(z) + \phi_2(z) + \cdots + \phi_n(z), \quad n = 1, 2, \ldots.$$

Then

$$\begin{aligned} f(z) - f_n(z) &= (\phi_1(z) + \phi_2(z) + \cdots) - (\phi_1(z) + \phi_2(z) + \cdots + \phi_n(z)) \\ &= \phi_{n+1}(z) + \phi_{n+2}(z) + \cdots. \end{aligned}$$

It follows that, for $z \in E$,

$$\begin{aligned} |f(z) - f_n(z)| &= |\phi_{n+1}(z) + \phi_{n+2}(z) + \cdots| \\ &= \left| \sum_{k=n+1}^{\infty} \phi_k(z) \right| \\ &\leq \sum_{k=n+1}^{\infty} |\phi_k(z)| \\ &\leq \sum_{k=n+1}^{\infty} M_k, \end{aligned} \tag{5.3}$$

where we have applied the Triangle Inequality for series (Theorem 1.8 of Unit B3) to obtain the second-to-last line. By assumption 2,

$$\sum_{k=n+1}^{\infty} M_k = \sum_{k=1}^{\infty} M_k - \sum_{k=1}^{n} M_k \to 0 \text{ as } n \to \infty.$$

Therefore the uniform convergence of $\sum_{n=1}^{\infty} \phi_n$ follows from inequality (5.3) and the strategy for proving uniform convergence. ■

Example 5.2

Prove that the series

$$\sum_{n=0}^{\infty} z^n$$

is uniformly convergent on $E = \{z : |z| \leq r\}$, where $0 < r < 1$.

Solution

Here

$$\phi_n(z) = z^n, \quad n = 0, 1, 2, \dots,$$

and

$$|\phi_n(z)| = |z|^n \le r^n, \quad \text{for } z \in E.$$

Hence assumption 1 of the M-test holds with $M_n = r^n$, for $n = 0, 1, 2, \dots$. Since

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad \text{for } 0 < r < 1,$$

assumption 2 of the M-test also holds, so $\sum_{n=0}^{\infty} z^n$ is uniformly convergent on E.

Example 5.2 shows that the power series $\sum_{n=0}^{\infty} z^n$ is uniformly convergent on each closed disc lying inside its disc of convergence. In fact, this result holds for an arbitrary power series, as we now ask you to verify (for simplicity the power series is about $\alpha = 0$).

Exercise 5.2

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with disc of convergence $\{z : |z| < R\}$, where $R > 0$. Show that the power series is uniformly convergent on each closed disc $\{z : |z| \le r\}$, where $0 < r < R$.

(*Hint*: Use the fact that the power series converges absolutely at each point in its disc of convergence, by Theorem 2.1 of Unit B3.)

The next example shows that the series for the zeta function is uniformly convergent on each closed half-plane lying in $\{z : \operatorname{Re} z > 1\}$. We will use this result in the next subsection when we prove that the zeta function is analytic on $\{z : \operatorname{Re} z > 1\}$.

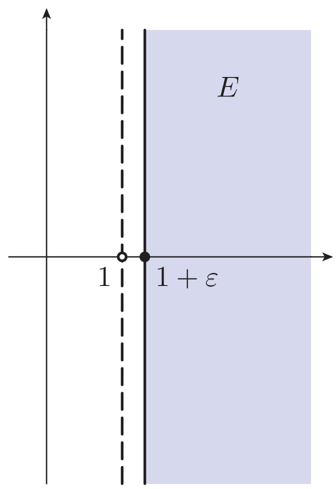


Figure 5.4 The set $E = \{z : \operatorname{Re} z \ge 1 + \varepsilon\}$

Example 5.3

Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ is uniformly convergent on the set

$$E = \{z : \operatorname{Re} z \ge 1 + \varepsilon\}, \quad \text{where } \varepsilon > 0,$$

which is illustrated in Figure 5.4.

Solution

Here

$$\phi_n(z) = \frac{1}{n^z} = e^{-z\log n}, \quad n = 1, 2, \ldots,$$

and

$$|\phi_n(z)| = e^{-(\operatorname{Re} z)\log n} = \frac{1}{n^{\operatorname{Re} z}} \le \frac{1}{n^{1+\varepsilon}}, \quad \text{for } z \in E.$$

Hence assumption 1 of the M-test holds with $M_n = 1/n^{1+\varepsilon}$, for $n = 1, 2, \ldots$. Since

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}$$

is convergent for $\varepsilon > 0$ (by Theorem 1.3 of Unit B3), assumption 2 of the M-test also holds, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ is uniformly convergent on E.

Exercise 5.3

Prove that the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

is uniformly convergent on $E = \{z : |z| \le 1\}$.

5.3 Weierstrass' Theorem

We now give a powerful result about uniformly convergent sequences of analytic functions. This result will enable us to prove that the zeta function is analytic and will also enable us to introduce another method for representing analytic functions, which will be discussed in Section 6.

Theorem 5.2 Weierstrass' Theorem

Let (f_n) be a sequence of functions, each of which is analytic on a region $\mathcal{R}$, and suppose that (f_n) converges uniformly to a function f on each closed disc in $\mathcal{R}$. Then

(a) f is analytic on $\mathcal{R}$

(b) the sequence (f'_n) converges uniformly to f' on each closed disc in $\mathcal{R}$.

Remarks

1. The hypothesis that (f_n) converges uniformly on 'each closed disc in $\mathcal{R}$' may seem strange. However, in practice it often happens that (f_n) converges uniformly to f on each closed disc in $\mathcal{R}$, and yet (f_n) does not converge uniformly to f on the whole of $\mathcal{R}$ (see Example 5.4, below). When applying the theorem, make sure you really do check uniform convergence on every closed disc in $\mathcal{R}$, not just on a selection of them.
2. Notice that if (f_n') converges uniformly to f' on each closed disc in $\mathcal{R}$, then (f_n') converges pointwise to f' on the whole of $\mathcal{R}$.
3. Having applied Weierstrass' Theorem to a suitable sequence (f_n), we can then apply it to the sequence (f_n') and hence deduce that (f_n'') converges uniformly to f'' on each closed disc in $\mathcal{R}$, and so on.

The proof of this result uses many earlier results and techniques, such as contour integration, Cauchy's Theorem, Morera's Theorem, Cauchy's First Derivative Formula, and the ubiquitous Estimation Theorem. Before giving the proof, we use Theorem 5.2 to prove that the zeta function is analytic.

Example 5.4

Prove that the zeta function ζ is analytic on $\{z : \operatorname{Re} z > 1\}$, and obtain a formula for $\zeta'(z)$.

Solution

In Example 5.3 we saw that the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is uniformly convergent on any closed half-plane $\{z : \operatorname{Re} z \geq 1 + \varepsilon\}$, for $\varepsilon > 0$. Since any closed disc in $\{z : \operatorname{Re} z > 1\}$ is contained in some such closed half-plane, we see that the series is uniformly convergent on each closed disc in $\{z : \operatorname{Re} z > 1\}$.

Next, the partial sum functions

$$\zeta_n(z) = 1 + \frac{1}{2^z} + \cdots + \frac{1}{n^z} = 1 + e^{-z\log 2} + \cdots + e^{-z\log n},$$

for $n = 1, 2, \ldots$, are each analytic on $\mathbb{C}$, so it follows from Weierstrass' Theorem that ζ is analytic on $\{z : \operatorname{Re} z > 1\}$.

To obtain a formula for ζ' we note from Remark 2 above that ζ_n' converges pointwise to ζ' on $\{z : \operatorname{Re} z > 1\}$. Now,

$$\begin{aligned}\zeta_n'(z) &= 0 - (\log 2)e^{-z\log 2} - \cdots - (\log n)e^{-z\log n}\\ &= -\left(\frac{\log 2}{2^z} + \cdots + \frac{\log n}{n^z}\right),\end{aligned}$$

so

$$\zeta'(z) = -\sum_{n=2}^{\infty} \frac{\log n}{n^z}, \quad \text{for } \operatorname{Re} z > 1.$$

Remarks

1. As indicated by this example, when Weierstrass' Theorem is used to prove that a function f defined by a series is analytic, the derivative f' can be obtained by term-by-term differentiation of the series.
2. In Subsection 6.2 you will see that the zeta function can be analytically continued to the region $\mathbb{C} - \{1\}$, and that it has a simple pole at the point 1.

Exercise 5.4

Obtain a formula for $\zeta''(z)$.

Proof of Weierstrass' Theorem We will prove Weierstrass' Theorem under the stronger assumption that the sequence (f_n) converges uniformly to f on the whole of $\mathcal{R}$. The (more general) version stated in Theorem 5.2 can then be deduced by applying this special case to open discs whose closures lie in $\mathcal{R}$.

The proof is in four steps.

1. To prove that f is continuous at each point $\alpha \in \mathcal{R}$, we need to show that for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $z \in \mathcal{R}$, then
$$|z-\alpha| < \delta \implies |f(z) - f(\alpha)| < \varepsilon.$$
Let us suppose, then, that $\varepsilon > 0$. By the uniform convergence of (f_n) to f, we can choose N such that
$$|f_n(z) - f(z)| < \tfrac{1}{3}\varepsilon, \quad \text{for all } n > N \text{ and all } z \in \mathcal{R}.$$
In particular,
$$|f_n(\alpha) - f(\alpha)| < \tfrac{1}{3}\varepsilon, \quad \text{for all } n > N.$$
Now we choose some fixed $n > N$. Then, since the function f_n is continuous at α, we can choose $\delta > 0$ such that if $z \in \mathcal{R}$, then
$$|z-\alpha| < \delta \implies |f_n(z) - f_n(\alpha)| < \tfrac{1}{3}\varepsilon.$$
It follows by the Triangle Inequality that, for $z \in \mathcal{R}$ and $|z-\alpha| < \delta$,
$$\begin{aligned} |f(z) - f(\alpha)| &= |(f(z) - f_n(z)) + (f_n(z) - f_n(\alpha)) + (f_n(\alpha) - f(\alpha))| \\ &\le |f(z) - f_n(z)| + |f_n(z) - f_n(\alpha)| + |f_n(\alpha) - f(\alpha)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon, \end{aligned}$$
as required. Hence f is continuous at α.
2. Next we show that if Γ is a contour in $\mathcal{R}$, then
$$\lim_{n\to\infty} \int_\Gamma f_n(z)\,dz = \int_\Gamma f(z)\,dz. \tag{5.4}$$

(Note that the integral on the right *is* defined, by step 1.) By the uniform convergence of (f_n) to f, for each $\varepsilon > 0$, there is an integer N such that

$$|f_n(z) - f(z)| < \varepsilon, \quad \text{for all } n > N \text{ and all } z \in \mathcal{R}.$$

Let L be the length of Γ. If $n > N$, then

$$\left| \int_\Gamma f_n(z)\,dz - \int_\Gamma f(z)\,dz \right| = \left| \int_\Gamma (f_n(z) - f(z))\,dz \right| \leq \varepsilon L,$$

by the Estimation Theorem. Since this holds for any $\varepsilon > 0$, equation (5.4) follows.

3. Now we show that f is analytic on each open disc D in $\mathcal{R}$ and hence throughout $\mathcal{R}$. Let Γ be any rectangular contour in D. Then, by step 2 and Cauchy's Theorem,

$$\int_\Gamma f(z)\,dz = \lim_{n\to\infty} \int_\Gamma f_n(z)\,dz = 0,$$

since each of the functions f_n is analytic on D. Hence, by Morera's Theorem (Theorem 5.4 of Unit B2), f itself is analytic on D, as required.

4. Finally, we show that (f_n') converges uniformly to f' on each closed disc in $\mathcal{R}$. Let $E = \{z : |z - \alpha| \leq r\}$ be a closed disc in $\mathcal{R}$, and choose $\rho > r$ such that the circle C with centre α and radius ρ and its inside lie in $\mathcal{R}$ (see Figure 5.5).

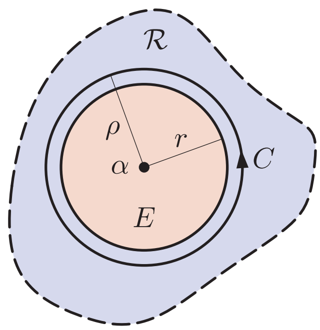


Figure 5.5 Disc E inside the circle C in $\mathcal{R}$

Then, for $z \in E$, we can use Cauchy's First Derivative Formula (Theorem 3.1 of Unit B2) to write

$$f_n'(z) - f'(z) = \frac{1}{2\pi i} \int_C \frac{f_n(w) - f(w)}{(w - z)^2}\,dw$$

(where $C = \{w : |w - \alpha| = \rho\}$). Now, (f_n) converges uniformly to f on $\mathcal{R}$, and hence also on C. Thus, given $\varepsilon > 0$, we can choose an integer N such that

$$|f_n(w) - f(w)| < \varepsilon, \quad \text{for all } n > N \text{ and all } w \in C.$$

Also, for $w \in C$ and $z \in E$,

$$|w - z| = |(w - \alpha) + (\alpha - z)| \geq |w - \alpha| - |z - \alpha| \geq \rho - r,$$

by the backwards form of the Triangle Inequality. Hence, by the Estimation Theorem,

$$\begin{aligned} |f_n'(z) - f'(z)| &= \left| \frac{1}{2\pi i} \int_C \frac{f_n(w) - f(w)}{(w - z)^2}\,dw \right| \\ &\leq \frac{1}{2\pi} \times \frac{\varepsilon}{(\rho - r)^2} \times 2\pi\rho = K\varepsilon, \end{aligned}$$

for all $n > N$, where $K = \rho/(\rho - r)^2$. Since this holds for any $\varepsilon > 0$, we see that (f_n') converges uniformly to f' on E. ■

Further exercises

Exercise 5.5

Prove that the sequence of functions

$$f_n(z) = z + \frac{z^2}{n}, \quad n = 1, 2, \ldots,$$

is uniformly convergent on each closed disc of the form $\{z : |z| \leq r\}$, where $r > 0$.

Exercise 5.6

Prove that the series

$$\sum_{n=1}^{\infty} \frac{z^n}{1+z^n}$$

is uniformly convergent on each closed disc of the form $E = \{z : |z| \leq r\}$, where $0 < r < 1$. Deduce that the sum function f is analytic on $\{z : |z| < 1\}$, and write down a formula for f'.

History of Weierstrass' Theorem

Weierstrass was responsible for placing the concept of uniform convergence, and its significant implications, on a sound mathematical basis. Weierstrass' Theorem was published in a paper from 1880, and the M-test also appeared in that work, as a footnote. It took the mathematical community some time to appreciate Weierstrass' rigorous approach to sequences of functions (which he had been developing even before 1880). In some satisfaction, Weierstrass wrote to Schwarz in 1881:

> My latest paper created more of a sensation among the French than it really deserves; people seem finally to realise the significance of the concept of uniform convergence.
>
> (Gray, 2015, p. 210)

6 Special functions

After working through this section, you should be able to:

- define the *gamma function* as an improper integral
- use properties of the gamma function, such as its functional equation
- understand how the gamma function can be analytically continued
- appreciate the Prime Number Theorem and the Riemann Hypothesis.

In Unit A2 we introduced some of the 'elementary' functions of complex analysis, namely polynomial functions, rational functions, the exponential function, trigonometric functions and hyperbolic functions, and some of their inverse functions. However, there are many other functions that have been given names in mathematics and physics because of their importance. Such functions are loosely termed **special functions**, and they are often denoted by Greek letters.

We met one example of a special function in the previous section, the zeta function, and in this section we will meet another, the gamma function. We will also discuss important applications of these two complex functions to number theory.

6.1 The gamma function

In the eighteenth century, mathematicians investigated the problem of whether there is a differentiable real function f with the property that $f(n) = 1 \times 2 \times \cdots \times n = n!$, for each positive integer n. The question was answered in the positive by Euler in 1729. The function he discovered, now known as the *gamma function*, proved to be a complex differentiable function. The definition that we give is equivalent to (although slightly different from) one of Euler's original definitions.

Definition

The **gamma function** is

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad (\operatorname{Re} z > 1).$$

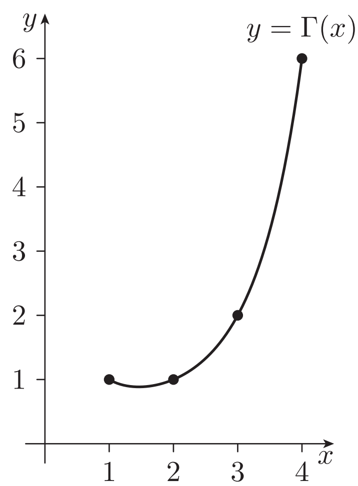


Figure 6.1 Graph of $y = \Gamma(x)$

We will see shortly that the gamma function satisfies $\Gamma(n) = (n-1)!$, for each positive integer n, so the function $f(z) = z\Gamma(z)$ satisfies $f(n) = n!$. The equation $\Gamma(n) = (n-1)!$ is illustrated in the graph of $y = \Gamma(x)$ for *real* values x (between 1 and 4) shown in Figure 6.1.

For each value of z with $\operatorname{Re} z > 1$, the image $\Gamma(z)$ is defined as an improper integral. It can be shown that this improper integral exists. Furthermore, the next theorem says that Γ is analytic on its domain $\{z : \operatorname{Re} z > 1\}$.

Theorem 6.1

The gamma function is an analytic function and

$$\Gamma'(z) = \int_0^\infty e^{-t} t^{z-1} \log t \, dt \quad (\operatorname{Re} z > 1).$$

Observe that $e^{-t}t^{z-1} = e^{-t}e^{(z-1)\log t}$. Hence

$$\frac{d}{dz}(e^{-t}t^{z-1}) = e^{-t}e^{(z-1)\log t} \log t = e^{-t}t^{z-1} \log t.$$

Thus we see that $\Gamma'(z)$ is obtained from $\Gamma(z)$ simply by differentiating the integrand $e^{-t}t^{z-1}$ of $\Gamma(z)$ with respect to z.

This sort of trick (differentiating 'under the integral sign') is a valid mathematical procedure under suitable conditions, and it is possible to verify those conditions, and prove the theorem, using the techniques that you have learned so far. However, the details of this verification are lengthy, so we choose to omit the proof of the theorem, and instead move on to look at other properties of the gamma function.

One of the most important properties is encapsulated in the following theorem.

Theorem 6.2

$$\Gamma(z+1) = z\Gamma(z), \quad \text{for } \operatorname{Re} z > 1.$$

The equation $\Gamma(z+1) = z\Gamma(z)$ is referred to as the **functional equation for the gamma function**.

Proof The integral for $\Gamma(z)$ is 'improper at ∞', and it is also 'improper at 0', because the integrand $e^{-t}t^{z-1}$ is not in general defined at $t = 0$ (unless $z-1$ is a positive integer, or the reciprocal of a positive integer). The integral can be evaluated by taking two limits as follows:

$$\int_0^\infty e^{-t}t^{z-1}\, dt = \lim_{r\to\infty} \lim_{\varepsilon\to 0} \int_\varepsilon^r e^{-t}t^{z-1}\, dt,$$

where the limit $\lim\limits_{\varepsilon\to 0}$ is taken through positive values of ε.

Let $\operatorname{Re} z > 1$. For $\varepsilon > 0$ and $r > 0$, we can integrate by parts (using the fact that the derivative of t^z with respect to t is zt^{z-1}) to give

$$\begin{aligned}\int_\varepsilon^r e^{-t}t^z\, dt &= \left[-e^{-t}t^z\right]_\varepsilon^r + z\int_\varepsilon^r e^{-t}t^{z-1}\, dt \\ &= -e^{-r}r^z + e^{-\varepsilon}\varepsilon^z + z\int_\varepsilon^r e^{-t}t^{z-1}\, dt. \qquad (6.1)\end{aligned}$$

Let us prove that

$$e^{-\varepsilon}\varepsilon^z \to 0 \text{ as } \varepsilon \to 0 \quad \text{and} \quad e^{-r}r^z \to 0 \text{ as } r \to \infty.$$

We write $z = x + iy$, with $x > 1$. For the first limit, we have, for $\varepsilon > 0$,

$$|e^{-\varepsilon}\varepsilon^z| = e^{-\varepsilon}\varepsilon^x \to 1 \times 0 = 0 \text{ as } \varepsilon \to 0.$$

For the second limit, choose a fixed integer m greater than x, and observe that, for $r > 0$,

$$e^r = 1 + r + \frac{r^2}{2!} + \cdots > \frac{r^{m+1}}{(m+1)!}.$$

Then, for $r > 0$,

$$|-e^{-r}r^z| = \frac{r^x}{e^r} < \frac{r^m}{r^{m+1}/(m+1)!} = \frac{(m+1)!}{r}.$$

Since $(m+1)!/r \to 0$ as $r \to \infty$, it follows that $e^{-r}r^z \to 0$ as $r \to \infty$.

Taking the limit as $\varepsilon \to 0$ and $r \to \infty$ in equation (6.1), we obtain

$$\int_0^\infty e^{-t}t^z\,dt = z\int_0^\infty e^{-t}t^{z-1}\,dt,$$

that is, $\Gamma(z+1) = z\Gamma(z)$, for $\operatorname{Re} z > 1$, as required. ∎

Corollary

$$\Gamma(n+1) = n!, \quad \text{for } n = 1, 2, \ldots.$$

Proof For $r > 0$, we have that

$$\int_0^r e^{-t}t\,dt = \left[-e^{-t}t\right]_0^r + \int_0^r e^{-t}\,dt = -re^{-r} + (1 - e^{-r}) \to 1$$

as $r \to \infty$. It follows that $\Gamma(2) = 1 = 1!$. If $n > 1$, then, by applying Theorem 6.2 repeatedly, we see that

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n)\\ &= n(n-1)\Gamma(n-1)\\ &= \cdots\\ &= n(n-1)\cdots 2\Gamma(2) = n!,\end{aligned}$$

as required. ∎

Exercise 6.1

Prove the following formulas for the gamma function.

(a) $\Gamma(z) = n^z \displaystyle\int_0^\infty e^{-nt}t^{z-1}\,dt, \quad$ for $n = 1, 2, \ldots$ and $\operatorname{Re} z > 1$

(b) $\Gamma(z) = \displaystyle\int_0^1 \left(\log\frac{1}{t}\right)^{z-1} dt, \quad$ for $\operatorname{Re} z > 1$

(*Hint*: For part (b), use the substitution $t = -\log u$.)

The formula in Exercise 6.1(b) gives the form in which the gamma function was first introduced by Euler.

Theorem 6.3

The gamma function has an analytic continuation Γ to $\mathbb{C} - \{0, -1, -2, \ldots\}$ with simple poles at $0, -1, -2, \ldots$ such that

$$\operatorname{Res}(\Gamma, -k) = \frac{(-1)^k}{k!}, \quad \text{for } k = 0, 1, 2, \ldots.$$

The functional equation for the gamma function holds on $\mathbb{C} - \{0, -1, -2, \ldots\}$.

We use the same letter Γ and the same name 'gamma function' for the analytic continuation of the gamma function.

Proof The key to obtaining the desired analytic continuation lies in rewriting the functional equation for the gamma function in the form

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad \text{for } \operatorname{Re} z > 1. \tag{6.2}$$

If we now define the function Γ_0 by

$$\Gamma_0(z) = \frac{\Gamma(z+1)}{z} \quad (\operatorname{Re} z > 0),$$

then Γ_0 is analytic on $\{z : \operatorname{Re} z > 0\}$ and agrees with Γ on $\{z : \operatorname{Re} z > 1\}$, by equation (6.2). Thus Γ_0 is an analytic continuation of Γ to $\{z : \operatorname{Re} z > 0\}$, which we promptly rename Γ in accordance with the remark above. The process is now repeated, defining the function Γ_1 by

$$\Gamma_1(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} \quad (\operatorname{Re} z > -1,\ z \neq 0),$$

so Γ_1 is analytic on $\{z : \operatorname{Re} z > -1,\ z \neq 0\}$ and agrees with Γ on $\{z : \operatorname{Re} z > 0\}$. Since $\Gamma(1) = \Gamma(2)/1 = 1 \neq 0$, we deduce that Γ_1 has a simple pole at 0. Renaming Γ_1 as Γ and continuing the process indefinitely, we find that Γ can be analytically continued to $\mathbb{C} - \{0, -1, -2, \ldots\}$ in such a way that the functional equation holds on this set (see Figure 6.2).

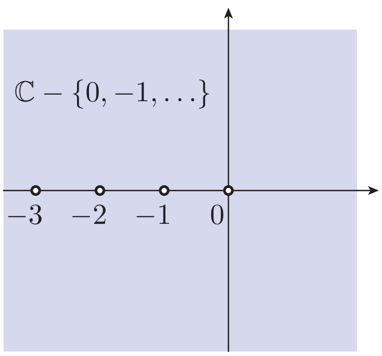


Figure 6.2 The set $\mathbb{C} - \{0, -1, -2, \ldots\}$

To find the residue of Γ at $-k$, for $k = 0, 1, 2, \ldots$, we note that

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \cdots = \frac{\Gamma(z+k+1)}{z(z+1)\cdots(z+k)},$$

for $z \neq 0, -1, -2, \ldots$. Then, for $k = 0, 1, 2, \ldots$, since Γ is continuous at 1, we see that

$$\begin{aligned}\lim_{z\to -k}(z+k)\Gamma(z) &= \lim_{z\to -k} \frac{\Gamma(z+k+1)}{z(z+1)\cdots(z+k-1)}\\ &= \frac{\Gamma(1)}{(-k)(-k+1)\cdots(-1)}\\ &= \frac{(-1)^k}{k!},\end{aligned}$$

so Γ has a simple pole at $-k$ with $\operatorname{Res}(\Gamma, -k) = (-1)^k/k!$, by Theorem 1.2 of Unit C1. ■

The graph of the *real* function $x \longmapsto \Gamma(x)$ $(x \in \mathbb{R})$ is shown in Figure 6.3. The graph shows clearly that Γ has singularities at $0, -1, -2, -3, \ldots$.

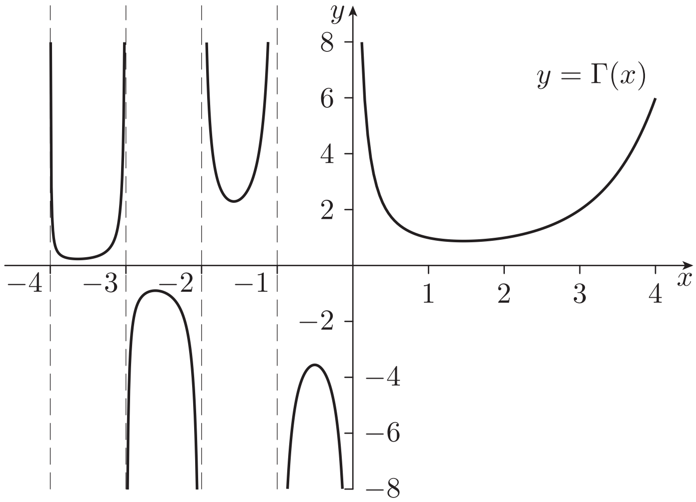


Figure 6.3 Graph of $y = \Gamma(x)$

Figure 6.4 is a graph of the surface $s = |\Gamma(z)|$. The poles of Γ are seen as 'spikes' arranged in a line along the negative real axis.

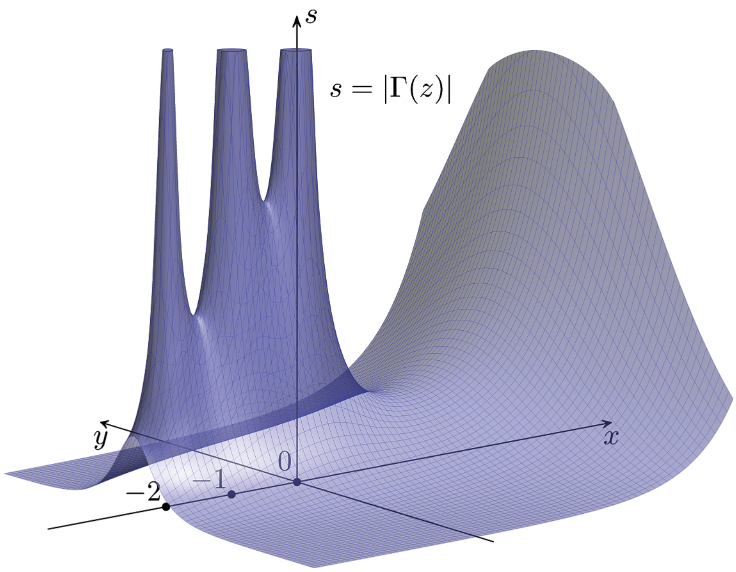


Figure 6.4 Graph of $s = |\Gamma(z)|$

Let us now discuss some methods for calculating values of the gamma function. To facilitate this discussion we first evaluate the following well-known integral. There are many ways of evaluating this integral, including some cunning applications of contour integrals and the Residue Theorem. However, we sketch a neater and more memorable proof that uses the theory of *double integrals*. Do not be concerned if you are unfamiliar with this theory and cannot follow aspects of the proof – it will not be assessed.

Theorem 6.4

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Proof Observe that

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy.\end{aligned}$$

Now we make the substitution $x = r\cos\theta$ and $y = r\sin\theta$ (and replace $dx\,dy$ by $r\,d\theta\,dr$) to switch from Cartesian to polar coordinates. Since $x^2 + y^2 = r^2$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy &= \int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2}r\,d\theta\,dr\\ &= 2\pi\int_{0}^{\infty} re^{-r^2}\,dr\\ &= 2\pi\left[-\tfrac{1}{2}e^{-r^2}\right]_0^{\infty} = \pi.\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

■

Gaussian integral

The result of Theorem 6.4 was first discovered in 1780 by the French mathematician and scientist Pierre-Simon Laplace, whom you met in Unit A4. In fact, Laplace found several different techniques for evaluating this and other remarkable integrals. One technique that he favoured was to make substitutions involving complex numbers, allowing him to take the 'passage from the real to the imaginary', as he put it.

Today the integral $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$ is often called the **Gaussian integral**, named after the German mathematician Carl Friedrich Gauss, whom you first encountered in Unit A1. Gauss observed the fundamental importance of the integral in statistics. Indeed, the function

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

is the probability density function for the standard normal distribution, with mean 0 and standard deviation 1. The factor $\dfrac{1}{\sqrt{2\pi}}$

is needed in order to ensure that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

The Gaussian integral cannot be evaluated using the Fundamental Theorem of Calculus because there is no way to combine elementary functions to give a function with derivative e^{-x^2}. Joseph Liouville, whom you met in Unit B2, developed a system for determining whether any given function can be integrated to give a primitive that can be expressed as a combination of elementary functions. This procedure was largely forgotten until 1969, when the American mathematician Robert Henry Risch (1939–) published an algorithm for implementing Liouville's ideas. The algorithm produces the desired primitive if this is possible, and it shows that no such primitive exists otherwise. Today it is widely used for symbolic integration in computer algorithms.

We can use Theorem 6.4, and the functional equation for Γ, to evaluate Γ at $1/2$.

Theorem 6.5

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$

Proof By definition,

$$\Gamma\left(\tfrac{3}{2}\right) = \int_0^{\infty} e^{-t} t^{1/2}\,dt.$$

On substituting $t = x^2$, $dt = 2x\,dx$, we obtain

$$\Gamma\left(\tfrac{3}{2}\right) = \int_0^{\infty} e^{-x^2} \times x \times 2x\,dx = \int_0^{\infty} 2xe^{-x^2} \times x\,dx.$$

We now integrate by parts using the fact that the derivative of $-e^{-x^2}$ with respect to x is $2xe^{-x^2}$, to give

$$\Gamma\left(\tfrac{3}{2}\right) = \lim_{r\to\infty}\left[-xe^{-x^2}\right]_0^r + \int_0^{\infty} e^{-x^2}\,dx.$$

The limit on the left is 0, and the integral on the right is half the integral of Theorem 6.4 (because $x \longmapsto e^{-x^2}$ is an even function). Hence

$$\Gamma\left(\tfrac{3}{2}\right) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Therefore, by the functional equation for the gamma function,

$$\Gamma\left(\tfrac{1}{2}\right) = \frac{\Gamma\left(\tfrac{3}{2}\right)}{\tfrac{1}{2}} = \sqrt{\pi},$$

as required ■

As you have seen, in calculating $\Gamma\left(\frac{1}{2}\right)$ we also calculated $\Gamma\left(\frac{3}{2}\right)$. Try evaluating the following further values of Γ by using Theorem 6.2.

Exercise 6.2

Evaluate

(a) $\Gamma\left(\frac{5}{2}\right)$, (b) $\Gamma\left(-\frac{1}{2}\right)$.

To finish this subsection we briefly discuss some of the many attractive mathematical properties enjoyed by the gamma function. *This material will not be assessed.*

There is an alternative way to define the gamma function, as an infinite product, as follows:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}, \quad \text{for } z \in \mathbb{C} - \{0, -1, -2, \ldots\}. \qquad (6.3)$$

This formula has the advantage over the integral definition that it is valid for all complex numbers z, apart from the poles of Γ, and it can be used to calculate approximate values of Γ.

For large *positive* values x, there is another useful formula for the gamma function, which can be used to calculate approximate values of Γ. It says that

$$\Gamma(x) \approx \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x.$$

You can see from this formula that $\Gamma(x)$ grows extremely quickly as x increases through positive values.

If x is equal to a positive integer n, then $\Gamma(x) = (n-1)!$, so, for large n,

$$n! = n\Gamma(n) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

This approximation to $n!$ is known as **Stirling's Formula**, named after the Scottish mathematician James Stirling (1692–1770).

Finally, we give just one of the many elegant identities satisfied by the gamma function:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad \text{for } z \in \mathbb{C} - \mathbb{Z}.$$

This identity is useful for calculating values of Γ. For example, set $z = \frac{1}{2}$ to see that

$$\Gamma\left(\tfrac{1}{2}\right)^2 = \frac{\pi}{1} = \pi, \quad \text{so} \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi},$$

as we know already from Theorem 6.5.

Origin of the gamma function

As we have seen, the problem of extending the factorial function of the positive integers to a well-behaved function of the real numbers was first solved by Euler. His work was developed by Gauss, who defined a *complex* function Π that coincides with the factorial function on the positive integers. In a letter to the German mathematician Friedrich Wilhelm Bessel (1784–1846), Gauss wrote:

> But if one doesn't want ... countless fallacies and paradoxes and contradictions to be exposed, $1 \cdot 2 \cdot 3 \cdots z$ must not be used as the definition of $\Pi(z)$, since such a definition has a precise meaning only when z is an integer; rather, one must start with a definition of greater generality, applicable even to imaginary values of z, of which that one occurs as a special case. I have chosen the following
>
> $$\Pi(z) = \frac{(1 \cdot 2 \cdot 3 \cdots k)k^z}{(z+1)(z+2)(z+3)\cdots(z+k)}$$
>
> when k becomes infinite.
>
> (Remmert, 1998, p. 34)

Gauss's formula is of a similar form to equation (6.3). With his notation, however, we have the convenient formula $\Pi(n) = n!$. But the French mathematician Adrien-Marie Legendre (1752–1833) introduced the now-standard notation Γ, which satisfies $\Gamma(z+1) = \Pi(z)$ (and hence $\Gamma(n) = (n-1)!$).

6.2 The Riemann Hypothesis

This subsection is intended for reading only (it will not be assessed).

As long ago as 1740, Euler was aware of the connection between the zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots \quad (\operatorname{Re} z > 1)$$

(which Euler considered for real values of z only) and the sequence of prime numbers:

$$2, 3, 5, 7, 11, 13, 17, 19, \ldots.$$

To see this connection, notice that

$$\frac{1}{2^z}\zeta(z) = \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \cdots,$$

so

$$\left(1 - \frac{1}{2^z}\right)\zeta(z) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \cdots.$$

Similarly,

$$\left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\zeta(z) = 1+\frac{1}{5^z}+\frac{1}{7^z}+\frac{1}{11^z}+\frac{1}{13^z}+\cdots,$$

and continuing indefinitely we obtain

$$\left(\left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{5^z}\right)\cdots\right)\zeta(z) = 1,$$

which gives,

$$\zeta(z) = \left(\left(1-\frac{1}{2^z}\right)\left(1-\frac{1}{3^z}\right)\left(1-\frac{1}{5^z}\right)\cdots\right)^{-1}, \quad \text{for } \operatorname{Re} z > 1. \tag{6.4}$$

That is, $\zeta(z)$ is equal to the reciprocal of the infinite product of all factors of the form $(1 - 1/p^z)$, in which p is a prime number.

Riemann saw that formula (6.4) could be used to bring techniques and results from complex analysis to bear on a long-standing problem concerning the distribution of prime numbers. At the end of the eighteenth century, both Gauss and Legendre had observed, by calculation, that the sequence of prime numbers thins out in a way that, although locally erratic, can be described rather precisely in the long term. Their observation concerns the counting function

$$\pi(x) = \text{the number of primes less than } x,$$

where $x > 0$. Some values of $\pi(x)$ are recorded below, along with the corresponding values of $x/\log x$ (given to the nearest integer) for comparison.

x	10	10^2	10^3	10^6	10^9	10^{12}
$\pi(x)$	4	25	168	78 498	50 847 534	37 607 912 018
$x/\log x$	4	22	145	72 382	48 254 942	36 191 206 825

It appears from the table that

$$\pi(x) \approx x/\log x.$$

Indeed, close inspection of tables like this led to the conjecture that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1,$$

a result that is now called the Prime Number Theorem.

Some progress towards the Prime Number Theorem had been made by Riemann's time, but it was still far from being proved. In a remarkable paper 'On the number of primes less than a given magnitude' in 1859, Riemann showed that a proof of the Prime Number Theorem could be given if certain properties of the zeta function could be established. These properties have to do with the locations of the zeros of the function ζ. Now, equation (6.4) shows that the zeta function has no zeros for $\operatorname{Re} z > 1$, but Riemann found an analytic continuation of ζ to the whole of $\mathbb{C} - \{1\}$, and it was the zeros of this analytic continuation that concerned him.

To analytically continue ζ to $\mathbb{C} - \{1\}$, we define the function

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} \quad (\operatorname{Re} z > 0),$$

which can be shown to be analytic on $\{z : \operatorname{Re} z > 0\}$. For $\operatorname{Re} z > 1$,

$$\begin{aligned}\zeta(z) - \eta(z) &= \sum_{n=1}^{\infty} \frac{1}{n^z} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} \\ &= 2\sum_{n=1}^{\infty} \frac{1}{(2n)^z} \\ &= \frac{2}{2^z}\sum_{n=1}^{\infty} \frac{1}{n^z} = 2^{1-z}\zeta(z).\end{aligned}$$

That is,

$$\zeta(z) = \frac{\eta(z)}{1 - 2^{1-z}}.$$

Using this equation, we can analytically continue ζ to $\operatorname{Re} z > 0$, with a simple pole at 1 (because $1 - 2^{1-z} = 0$ when $z = 1$).

Next, Riemann established the **functional equation for the zeta function**

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2)\Gamma(1-z)\zeta(1-z), \quad \text{for } 0 < \operatorname{Re} z < 1.$$

We can use this equation to define $\zeta(z)$ for $\operatorname{Re} z < 1$, because $\operatorname{Re} z < 1$ if and only if $\operatorname{Re}(1-z) > 0$, so each of the terms on the right-hand side is analytic for $\operatorname{Re} z < 1$ (apart from a single, simple pole of $\zeta(1-z)$ at $z = 0$, which 'cancels' with a simple zero of $\sin(\pi z/2)$). We thereby obtain an analytic continuation of ζ to $\mathbb{C} - \{1\}$. The resulting function ζ has simple zeros at the negative even integers $-2, -4, -6, \ldots$, corresponding to zeros of $\sin(\pi z/2)$, which are called the **trivial zeros** of the zeta function (because we know all about them). Any other zeros of ζ must lie in $\{z : 0 \le \operatorname{Re} z \le 1\}$, the so-called **critical strip** (see Figure 6.5).

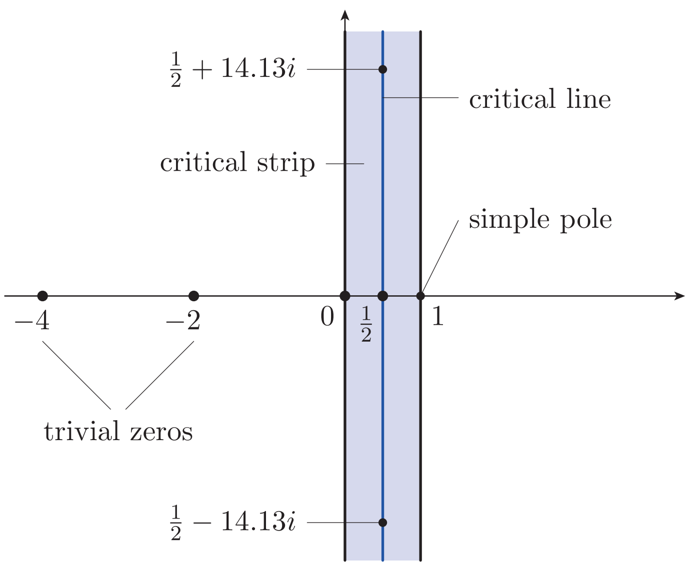


Figure 6.5 Zeros of the zeta function on and outside the critical strip

Riemann showed that the Prime Number Theorem could be proved if there are not too many zeros of ζ near the edges of the critical strip, and he made the startling conjecture in 1859 that in fact all the zeros of ζ in the critical strip lie on the **critical line** $\left\{z : \operatorname{Re} z = \frac{1}{2}\right\}$. Figure 6.5 shows the two zeros $\frac{1}{2} \pm 14.13i$ on the critical line (correct to two decimal places) that are closest to the real axis. This conjecture is now known as the Riemann Hypothesis, and it is still unresolved at the time of writing, despite the efforts of some of the best mathematicians since Riemann's day.

The Riemann Hypothesis is considered by some to be the most important unresolved problem in mathematics. It is one of the seven Millennium Prize Problems that were stated in the year 2000 by the Clay Mathematics Institute, a foundation for promoting mathematics. Each of the seven problems comes with an award of one million US dollars for anyone who finds a solution.

No one is sure on what evidence Riemann made his conjecture, but extensive numerical investigations have detected more than a billion zeros of ζ in the critical strip, *all* lying on $\left\{z : \operatorname{Re} z = \frac{1}{2}\right\}$. The first few of these, and the simple pole at $z = 1$, are visible in the plot of the surface $s = |\zeta(z)|$ in Figure 6.6.

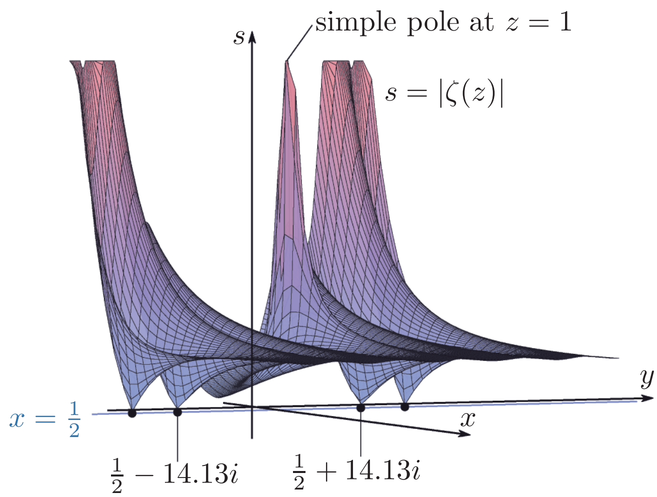


Figure 6.6 Graph of $s = |\zeta(z)|$

The English mathematician G.H. Hardy (1877–1947) proved in 1914 that infinitely many zeros of ζ lie on the critical line, but this does not prevent some others lying off it. Better still, it was later shown that, in a certain sense, at least two-fifths of the zeros in the critical strip lie on the critical line.

The Prime Number Theorem itself was proved independently in 1896 by the French mathematician Jacques Hadamard (1865–1963) and the Belgian mathematician Charles Jean de la Vallée Poussin (1866–1962), using the complex analytic approach suggested by Riemann, but without needing the full strength of the Riemann Hypothesis. Since then several other proofs have been found, including one that does not require complex analysis, but

the simplest so far was given in 1980 by the American mathematician Donald Newman (1930–2007). This proof again used the zeta function, but needed only the elementary fact that ζ is zero-free on $\{z : \operatorname{Re} z \geq 1\}$.

In spite of this, interest in the Riemann Hypothesis remains strong, partly because many results in number theory have been proved on the assumption that the Riemann Hypothesis is true. Proving such results may seem pointless if we do not know whether the Riemann Hypothesis is true, but there is an ulterior motive: should any of these results lead to a contradiction, then the Riemann Hypothesis would have to be false!

Further exercises

Exercise 6.3

The formula

$$V_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)} r^n, \quad n = 1, 2, \ldots,$$

gives the n-dimensional volume of the ball B of radius r in $\mathbb{R}^n$, given by

$$B = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq r^2\}.$$

Verify the following equations.

(a) $V_1 = 2r$ (b) $V_2 = \pi r^2$ (c) $V_3 = \frac{4}{3}\pi r^3$ (d) $V_4 = \frac{1}{2}\pi^2 r^4$

Exercise 6.4

A remarkable identity, due to Euler, is

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \text{for } \operatorname{Re}\alpha > 0 \text{ and } \operatorname{Re}\beta > 0.$$

Verify that the identity holds for $\alpha = \beta = \frac{1}{2}$ (by using the substitution $t = \sin^2\theta$ in the integral on the left).

(The function

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

is called the **beta function**. It plays an important role in the subject of analytic number theory.)

Solutions to exercises

Solution to Exercise 1.1

(a) Once: $2 \times (+1) + 1 \times (-1) = 1$

(b) Zero times

Solution to Exercise 1.2

(a) Since $|\gamma(t)| = t$, for $t \in [1,3]$, we have

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{\pi i t}, \quad \text{for } t \in [1,3].$$

Thus one choice of continuous argument function is

$$\theta(t) = \pi t \quad (t \in [1,3]).$$

(b) Each point $1 + it$, where $t \in [0,1]$, lies in the upper-right quadrant (or on the positive real line, when $t = 0$). Therefore

$$\operatorname{Arg}(1 + it) = \tan^{-1}\frac{t}{1} = \tan^{-1} t.$$

This is a continuous function of t, so we see that one choice of continuous argument function for Γ is

$$\theta(t) = \tan^{-1} t \quad (t \in [0,1]).$$

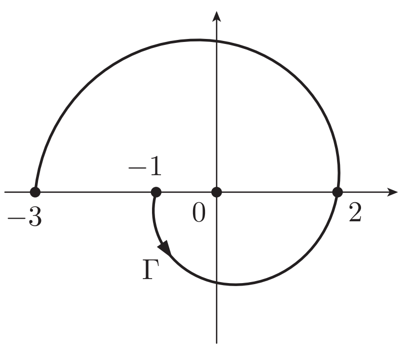
(c) Consider the continuous function

$$\theta(t) = -4t \quad (t \in [0, 2\pi]).$$

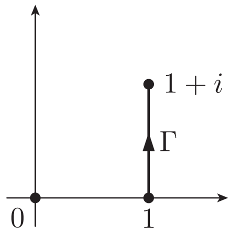
Since $\gamma(t) = e^{i\theta(t)}$, for $t \in [0, 2\pi]$, we see that θ is a continuous argument function for Γ.

Solution to Exercise 1.3

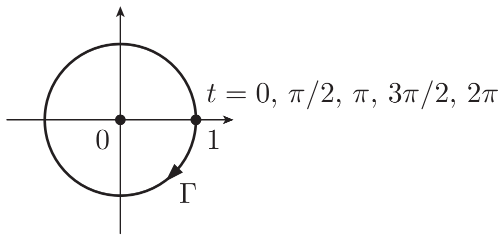
(a) $$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(3) - \theta(1)) = \frac{1}{2\pi}(3\pi - \pi) = 1$$



(b) $$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(1) - \theta(0)) = \frac{1}{2\pi}(\tan^{-1} 1 - \tan^{-1} 0) = \frac{1}{8}$$



(c) $$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(2\pi) - \theta(0)) = \frac{1}{2\pi}(-4 \times 2\pi - 0) = -4$$



(The circle is traversed four times clockwise.)

Solution to Exercise 1.4

Let one continuous argument function for Γ be

$$\theta\colon t \longmapsto \theta(t) \quad (t \in [a,b]).$$

Then

$$\theta\colon t \longmapsto \theta(t) \quad (t \in [a,c])$$

is a continuous argument function for Γ_1 and

$$\theta\colon t \longmapsto \theta(t) \quad (t \in [c,b])$$

is a continuous argument function for Γ_2.

Now,

$$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

$$\operatorname{Wnd}(\Gamma_1, 0) = \frac{1}{2\pi}(\theta(c) - \theta(a)),$$

$$\operatorname{Wnd}(\Gamma_2, 0) = \frac{1}{2\pi}(\theta(b) - \theta(c)).$$

Hence

$$\begin{aligned}
&\mathrm{Wnd}(\Gamma_1,0)+\mathrm{Wnd}(\Gamma_2,0)\\
&=\frac{1}{2\pi}(\theta(c)-\theta(a))+\frac{1}{2\pi}(\theta(b)-\theta(c))\\
&=\frac{1}{2\pi}(\theta(b)-\theta(a))\\
&=\mathrm{Wnd}(\Gamma,0),
\end{aligned}$$

as required.

Solution to Exercise 1.5

$\mathrm{Wnd}(\Gamma,1)=2,\quad \mathrm{Wnd}(\Gamma,-2)=1,$

$\mathrm{Wnd}(\Gamma,-2i)=1,\quad \mathrm{Wnd}(\Gamma,3i)=0.$

Solution to Exercise 1.6

Let $\gamma\colon [a,b]\longrightarrow\mathbb{C}$ be a parametrisation of the contour Γ.

Using Theorem 1.2 and the definition of a contour integral, we see that

$$\begin{aligned}
\mathrm{Wnd}(\Gamma,\alpha)&=\mathrm{Wnd}(\Gamma-\alpha,0)\\
&=\frac{1}{2\pi i}\int_{\Gamma-\alpha}\frac{1}{z}\,dz\\
&=\frac{1}{2\pi i}\int_a^b\frac{\gamma'(t)}{\gamma(t)-\alpha}\,dt\\
&=\frac{1}{2\pi i}\int_\Gamma\frac{1}{z-\alpha}\,dz,
\end{aligned}$$

as required.

Solution to Exercise 1.7

(a) $\mathrm{Wnd}(\Gamma,0)=2,\quad \mathrm{Wnd}(\Gamma,2)=1,$

$\mathrm{Wnd}(\Gamma,3i)=0.$

(b) Since

$$2+\cos\tfrac{3}{2}t\geq 2-1=1>0,\quad \text{for } 0\leq t\leq 4\pi,$$

we have

$$\frac{\gamma(t)}{|\gamma(t)|}=e^{it},\quad \text{for } 0\leq t\leq 4\pi.$$

Hence a continuous argument function for Γ is

$$\theta(t)=t\quad (t\in[0,4\pi]).$$

Thus

$$\mathrm{Wnd}(\Gamma,0)=\frac{1}{2\pi}(\theta(4\pi)-\theta(0))=2.$$

(c) By Exercise 1.6,

$$\int_\Gamma\frac{1}{z-2}\,dz=2\pi i\,\mathrm{Wnd}(\Gamma,2),$$

since Γ is closed and $2\notin\Gamma$. By part (a), $\mathrm{Wnd}(\Gamma,2)=1$, so

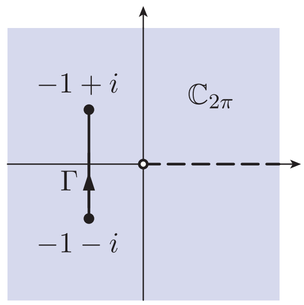
$$\int_\Gamma\frac{1}{z-2}\,dz=2\pi i\times 1=2\pi i.$$

Solution to Exercise 1.8

Observe that

$$\gamma(t)\in\mathbb{C}_{2\pi},\quad \text{for } -1\leq t\leq 1,$$

as shown in the figure.



Since $\mathrm{Arg}_{2\pi}$ is continuous on $\mathbb{C}_{2\pi}$, we deduce that

$$\theta(t)=\mathrm{Arg}_{2\pi}(\gamma(t))\quad (t\in[-1,1])$$

is a continuous argument function for Γ. Thus

$$\begin{aligned}
\mathrm{Wnd}(\Gamma,0)&=\frac{1}{2\pi}(\theta(1)-\theta(-1))\\
&=\frac{1}{2\pi}(\mathrm{Arg}_{2\pi}(-1+i)-\mathrm{Arg}_{2\pi}(-1-i))\\
&=\frac{1}{2\pi}\left(\frac{3\pi}{4}-\frac{5\pi}{4}\right)=-\frac{1}{4}.
\end{aligned}$$

Solution to Exercise 2.1

Since $f(z)=z^{10}(z-1)$, we see that

$$f'(z)=10z^9(z-1)+z^{10}.$$

Hence

$$\begin{aligned}
\frac{f'(z)}{f(z)}&=\frac{10z^9(z-1)+z^{10}}{z^{10}(z-1)}\\
&=\frac{10}{z}+\frac{1}{z-1}.
\end{aligned}$$

Thus f'/f has a simple pole at 0 with residue 10, and it has a simple pole at 1 with residue 1.

In comparison, f has a zero at 0 of order 10, and it has a zero at 1 of order 1.

Solution to Exercise 2.2

From Figure 2.3 it can be seen that

$$\mathrm{Wnd}(f(\Gamma), -1) = 3.$$

Hence the equation $f(z) = -1$ has exactly three solutions inside Γ.

Solution to Exercise 2.3

(a) On $\Gamma = \{z : |z| = 2\}$ a dominant term for f is $g(z) = z^5$, since, for $z \in \Gamma$,

$$\begin{aligned}|f(z) - g(z)| &= |z^3 + iz + 1| \\ &\leq |z|^3 + |z| + 1 \\ &= 11 < 32 = |g(z)|,\end{aligned}$$

by the Triangle Inequality.

Now, g has a zero of order five at 0, which is in $\{z : |z| < 2\}$ (and there are no other zeros). Since f and g are analytic on the simply connected region $\mathbb{C}$, and Γ is a simple-closed contour in $\mathbb{C}$, we can apply Rouché's Theorem to see that f has five zeros in $\{z : |z| < 2\}$.

(b) On $\Gamma = \{z : |z| = 1\}$ a dominant term for f is $g(z) = e^z$, since, for $z \in \Gamma$,

$$|f(z) - g(z)| = \left|-\tfrac{1}{3}z^4\right| = \tfrac{1}{3}$$

and

$$|g(z)| = |e^z| = e^{\operatorname{Re} z} \geq e^{-1} > \tfrac{1}{3},$$

so

$$|f(z) - g(z)| < |g(z)|.$$

Observe that g has no zeros in $\{z : |z| < 1\}$. Since f and g are analytic on the simply connected region $\mathbb{C}$, and Γ is a simple-closed contour in $\mathbb{C}$, we can apply Rouché's Theorem to see that f also has no zeros in $\{z : |z| < 1\}$.

(c) **(i)** On $\Gamma = \{z : |z| = 2\}$ a dominant term for f is $g(z) = z^5$, since, for $z \in \Gamma$,

$$\begin{aligned}|f(z) - g(z)| &= \left|-3z^3 - 1\right| \\ &\leq 3|z|^3 + 1 \\ &= 25 < 32 = |g(z)|,\end{aligned}$$

by the Triangle Inequality.

Now, g has a zero of order five at 0, which is in $\{z : |z| < 2\}$ (and there are no other zeros). Since f and g are analytic on the simply connected region $\mathbb{C}$, and Γ is a simple-closed contour in $\mathbb{C}$, we can apply Rouché's Theorem to see that f has five zeros in $\{z : |z| < 2\}$.

(ii) On $\Gamma = \{z : |z| = 1\}$ a dominant term for f is $g(z) = -3z^3$, since, for $z \in \Gamma$,

$$\begin{aligned}|f(z) - g(z)| &= |z^5 - 1| \\ &\leq |z|^5 + 1 \\ &= 2 < 3 = |g(z)|, \qquad \text{(S1)}\end{aligned}$$

by the Triangle Inequality.

Now, g has a zero of order three at 0, which is in $\{z : |z| < 1\}$ (and there are no other zeros). Since f and g are analytic on the simply connected region $\mathbb{C}$, and Γ is a simple-closed contour in $\mathbb{C}$, we can apply Rouché's Theorem to see that f has three zeros in $\{z : |z| < 1\}$.

(iii) Parts (i) and (ii) tell us that f has $5 - 3 = 2$ zeros in $\{z : 1 \leq |z| < 2\}$. Now observe that f has no zeros on $\{z : |z| = 1\}$ because setting $f(z) = 0$ in inequality (S1) gives $|g(z)| < |g(z)|$, which is impossible. It follows that f has two zeros in $\{z : 1 < |z| < 2\}$.

Solution to Exercise 2.4

(a) The Taylor series about 0 for exp is

$$\exp z = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots,$$

for $z \in \mathbb{C}$. Hence

$$\begin{aligned}|e^z - 1| &= \left|z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right| \\ &\leq |z| + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \cdots,\end{aligned}$$

by the Triangle Inequality for series (Theorem 1.8 of Unit B3). Thus, for $|z| \leq 1$,

$$|e^z - 1| \leq 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e^1 - 1 = e - 1.$$

(b) Let $\Gamma = \{z : |z| = 1\}$, and let f be the function

$$f(z) = e^z - 2z - 1.$$

On Γ a dominant term for f is $g(z) = -2z$, since

$$\begin{aligned}|f(z) - g(z)| &= |e^z - 1| \\ &\leq e - 1 \\ &< 2 = |g(z)|,\end{aligned}$$

for $z \in \Gamma$, where we have used part (a) to give the second-to-last line.

Now, f and g are entire, Γ is a simple-closed contour, and g has just one zero (a simple one at 0) inside Γ, so f has exactly one zero inside Γ, by Rouché's Theorem. Hence the equation $e^z = 2z + 1$ has exactly one solution in $\{z : |z| < 1\}$.

Solution to Exercise 2.5

Observe that $e^x + 3x^2 > 0$, for $x \in \mathbb{R}$, so $f(z) = e^z + 3z^2$ has no real zeros. Also,

$$\begin{aligned}\overline{f(z)} &= \overline{e^z + 3z^2} \\ &= \overline{e^z} + \overline{3z^2} \\ &= e^{\overline{z}} + 3\overline{z}^2 \\ &= f(\overline{z}),\end{aligned}$$

where we have used the identity $\overline{e^z} = e^{\overline{z}}$, from Exercise 4.11 of Unit A2.

Hence the two zeros of the function f in $\{z : |z| < 1\}$ are complex conjugates of one another.

Solution to Exercise 2.6

(a) The poles of f'/f are the zeros 1, 2 and 3 of f.

Since f has a zero of order three at the point 1, we deduce by Theorem 2.1 that f'/f has a simple pole at 1 with $\operatorname{Res}(f'/f, 1) = 3$. Similarly, f'/f has simple poles at 2 and 3 with

$$\operatorname{Res}(f'/f, 2) = 2 \quad \text{and} \quad \operatorname{Res}(f'/f, 3) = 1.$$

(b) By the Argument Principle, $\operatorname{Wnd}(f(\Gamma), 0)$ is the number of zeros of f inside Γ, counted according to their orders. Since f has $3 + 2 + 1 = 6$ zeros inside Γ, counted according to their orders, we have $\operatorname{Wnd}(f(\Gamma), 0) = 6$.

Solution to Exercise 2.7

We apply the corollary to the Argument Principle.

(a) Since $\operatorname{Wnd}\big(f(\Gamma), -\tfrac{1}{2}\big) = 1$, the equation

$$f(z) = -\tfrac{1}{2}$$

has one solution inside Γ.

(b) Since $\operatorname{Wnd}\big(f(\Gamma), \tfrac{1}{4}\big) = 2$, the equation

$$f(z) = \tfrac{1}{4}$$

has two solutions inside Γ.

(c) Since $\operatorname{Wnd}(f(\Gamma), 2i) = 0$, the equation

$$f(z) = 2i$$

has no solutions inside Γ.

Solution to Exercise 2.8

(a) **(i)** We let Γ_1 be the simple-closed contour $\partial S_1 = \{z : |z| = 2\}$ and choose a dominant term for f on Γ_1. If $g(z) = z^5$, then, for $z \in \Gamma_1$,

$$\begin{aligned}|f(z) - g(z)| &= |3z + 10| \\ &\le 3|z| + 10 \\ &\le 16,\end{aligned}$$

by the Triangle Inequality, and

$$|g(z)| = |z|^5 = 32 > 16.$$

Since f and g are entire, Rouché's Theorem implies that f has the same number of zeros as g inside Γ_1, namely five, arising from the zero of order five of g at 0.

(ii) We let Γ_2 be the simple-closed contour $\partial S_2 = \{z : |z| = 1\}$ and choose a dominant term for f on Γ_2. If $g(z) = 3z + 10$, then, for $z \in \Gamma_2$,

$$|f(z) - g(z)| = |z^5| = |z|^5 = 1$$

and

$$|g(z)| = |3z + 10| \ge \big||3z| - 10\big| = 7 > 1,$$

by the backwards form of the Triangle Inequality. Since f and g are entire, Rouché's Theorem implies that f has the same number of zeros as g inside Γ_2, namely none.

Remark: We could have chosen $g(z) = 10$ here. Then, as you can check, for $z \in \Gamma_2$,

$$|f(z) - g(z)| \le 4 \quad \text{and} \quad |g(z)| = 10 > 4.$$

Hence, by Rouché's Theorem, f has no zeros inside Γ_2 since $g(z) = 10$ has none.

(iii) Since f has five zeros in S_1, no zeros on Γ_2 (because, from part (ii), $|f(z) - g(z)| < |g(z)|$, for $z \in \Gamma_2$), and no zeros in S_2, we deduce that f has five zeros in S_3.

(iv) For real values x, the function

$$f(x) = x^5 + 3x + 10$$

is strictly increasing because

$$f'(x) = 5x^4 + 3 > 0, \quad \text{for } x \in \mathbb{R}.$$

Since $f(x) \to -\infty$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$, we see that f has exactly one real zero. Furthermore, since $\overline{f(z)} = f(\overline{z})$, we deduce that the other four zeros of f form two complex conjugate pairs. Hence f has two zeros in S_4.

(b) We let Γ be the simple-closed contour $\partial S = \left\{z : |z| = \frac{1}{2}\right\}$ and choose a dominant term for f on Γ. If $g(z) = 3z$, then, for $z \in \Gamma$,

$$|f(z) - g(z)| = |\operatorname{Log}(1+z)| \leq 2|z| = 1,$$

by the inequality given in the hint, and

$$|g(z)| = |3z| = \tfrac{3}{2} > 1.$$

Since f and g are both analytic on the open unit disc, which contains Γ, we can apply Rouché's Theorem to see that f has the same number of zeros as g inside Γ, namely one, arising from the simple zero of g at 0.

Solution to Exercise 3.1

Let f be an entire function that maps $\mathbb{C}$ into $\mathbb{R}$. Then $f(\mathbb{C})$ cannot possibly be open, because there are no open discs contained in $\mathbb{R}$. Hence, by the Open Mapping Theorem, the only possibility is that f is a constant function.

Solution to Exercise 3.2

(a) If $w \neq 1$, then $w - 1$ can be written in the form

$$w - 1 = \rho(\cos\phi + i\sin\phi),$$

where $\rho > 0$ and $-\pi < \phi \leq \pi$. Then, by Theorem 3.1 of Unit A1, the equation $1 + z^4 = w$ has exactly four distinct solutions

$$z_k = \rho^{1/4}\left(\cos\left(\tfrac{1}{4}\phi + \tfrac{1}{2}k\pi\right) + i\sin\left(\tfrac{1}{4}\phi + \tfrac{1}{2}k\pi\right)\right),$$

for $k = 0, 1, 2, 3$.

(b) The restriction of f to the sector

$$A = \{z : 0 < \operatorname{Arg} z < \pi/2\}$$

is one-to-one (because, for $w \neq 1$, at most one of the four solutions of $1 + z^4 = w$ can lie in A). Since

$$D = \{z : |z - 1 - i| < 1\} \subseteq A,$$

it follows that the restriction of f to D is one-to-one, as required.

Solution to Exercise 3.3

(a) Since the Taylor series about $\alpha = 0$ for f is

$$f(z) = z - z^2,$$

the function f is one-to-one near 0, by the Local Mapping Theorem.

If $\alpha = \frac{1}{2}$, then

$$f'(\alpha) = 1 - 2\alpha = 0,$$
$$f''(\alpha) = -2 \neq 0,$$

so f is two-to-one near $\frac{1}{2}$, by the corollary to the Local Mapping Theorem.

(b) If z_1, z_2 both lie in $D = \left\{z : |z| \leq \frac{1}{2}\right\}$, then

$$\begin{aligned} f(z_1) = f(z_2) &\implies z_1 - z_1^2 = z_2 - z_2^2 \\ &\implies z_1 - z_2 = z_1^2 - z_2^2 \\ &\implies z_1 - z_2 = (z_1 - z_2)(z_1 + z_2) \\ &\implies z_1 = z_2 \text{ or } z_1 + z_2 = 1. \end{aligned}$$

Now, $z_1 + z_2 = 1$ can happen for $z_1, z_2 \in D$ only if $z_1 = z_2 = \frac{1}{2}$ (since $\operatorname{Re}(z_1 + z_2) = \operatorname{Re} z_1 + \operatorname{Re} z_2 < 1$ otherwise). Hence the restriction of $f(z) = z - z^2$ to D is one-to-one.

If $r > \frac{1}{2}$, then the restriction of f to $\{z : |z| < r\}$ is not one-to-one, since f is two-to-one near $\frac{1}{2}$, by part (a).

Solution to Exercise 3.4

(a) To prove that $f(z) = \sin z$ is one-to-one on the region $\mathcal{R} = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$, note that if $z_1, z_2 \in \mathcal{R}$, then

$$\begin{aligned} &\sin z_1 = \sin z_2 \\ &\implies \frac{1}{2i}\left(e^{iz_1} - e^{-iz_1}\right) = \frac{1}{2i}\left(e^{iz_2} - e^{-iz_2}\right) \\ &\implies e^{iz_1} - e^{iz_2} = e^{-iz_1} - e^{-iz_2} \\ &\implies e^{iz_1} - e^{iz_2} = \frac{e^{iz_2} - e^{iz_1}}{e^{i(z_1+z_2)}} \\ &\implies e^{iz_1} = e^{iz_2} \text{ or } e^{i(z_1+z_2)} = -1. \end{aligned}$$

If $e^{iz_1} = e^{iz_2}$, then $iz_1 = iz_2 + 2n\pi i$, for some $n \in \mathbb{Z}$, so $z_1 = z_2$, since $|\operatorname{Re}(z_1 - z_2)| < \pi$.

If $e^{i(z_1+z_2)} = -1$, then $i(z_1 + z_2) = i\pi + 2n\pi i$, for some $n \in \mathbb{Z}$, which is impossible since $|\operatorname{Re}(z_1 + z_2)| < \pi$.

Hence, for $z_1, z_2 \in \mathcal{R}$,

$$\sin z_1 = \sin z_2 \implies z_1 = z_2,$$

so f is one-to-one on $\mathcal{R}$.

Since f is analytic on $\mathcal{R}$, we deduce, by the Inverse Function Rule, that f^{-1} is analytic on $f(\mathcal{R})$.

(b) It follows from Exercise 3.3(b) that $f(z) = z - z^2$ is one-to-one on $\{z : |z| < \frac{1}{2}\}$. Since f is analytic on $\mathcal{R}$, we deduce, by the Inverse Function Rule, that f^{-1} is analytic on $f(\mathcal{R})$.

Solution to Exercise 3.5

If f is analytic at α with $f'(\alpha) \neq 0$, then f is one-to-one near α, by the Local Mapping Theorem. It follows that there is a region $\mathcal{S}$, with $\alpha \in \mathcal{S}$, such that the restriction of f to $\mathcal{S}$ is one-to-one, so this restriction has an analytic inverse function, by the Inverse Function Rule.

Solution to Exercise 3.6

(a) The Taylor series about $\alpha = 0$ for $f(z) = e^z$ is

$$f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

Thus the Taylor series about $\beta = f(\alpha) = 1$ for f^{-1} is

$$f^{-1}(w) = b_0 + b_1(w-1) + b_2(w-1)^2 + b_3(w-1)^3 + \cdots,$$

where $b_0 = \alpha = 0$ and $b_1, b_2, b_3, \ldots$ satisfy

$$\begin{aligned} z = b_1\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) \\ + b_2\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)^2 \\ + b_3\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)^3 + \cdots. \end{aligned}$$

Equating coefficients of powers of z, we obtain

$$\begin{aligned} z:&\quad 1 = b_1 && \Longrightarrow b_1 = 1, \\ z^2:&\quad 0 = \frac{1}{2!}b_1 + b_2 && \Longrightarrow b_2 = -\frac{1}{2}b_1 = -\frac{1}{2}, \\ z^3:&\quad 0 = \frac{1}{3!}b_1 + \frac{2}{2!}b_2 + b_3 && \Longrightarrow b_3 = -\frac{1}{6}b_1 - b_2 = \frac{1}{3}. \end{aligned}$$

Hence

$$f^{-1}(w) = (w-1) - \frac{(w-1)^2}{2} + \frac{(w-1)^3}{3} - \cdots.$$

(This is the Taylor series about 1 for $\operatorname{Log} w$, as expected.)

(b) The Taylor series about $\alpha = 0$ for $f(z) = z - z^2$ is

$$f(z) = z - z^2.$$

Thus the Taylor series about $\beta = f(\alpha) = 0$ for f^{-1} is

$$f^{-1}(w) = b_0 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots,$$

where $b_0 = \alpha = 0$ and $b_1, b_2, b_3, \ldots$ satisfy

$$z = b_1(z - z^2) + b_2(z - z^2)^2 + b_3(z - z^2)^3 + \cdots.$$

Equating coefficients of powers of z, we obtain

$$\begin{aligned} z:&\quad 1 = b_1 && \Longrightarrow b_1 = 1, \\ z^2:&\quad 0 = -b_1 + b_2 && \Longrightarrow b_2 = b_1 = 1, \\ z^3:&\quad 0 = -2b_2 + b_3 && \Longrightarrow b_3 = 2b_2 = 2. \end{aligned}$$

Hence

$$f^{-1}(w) = w + w^2 + 2w^3 + \cdots.$$

Solution to Exercise 3.7

We are given that f maps the open unit disc D into the circle $\{w : |w| = \pi\}$. This set contains no open discs, so $f(D)$ cannot be open. The Open Mapping Theorem tells us that $f(D)$ *is* open if f is not constant, so we deduce that f must be constant.

Solution to Exercise 3.8

(a) The Taylor series about $\alpha = 0$ for f is

$$f(z) = \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots.$$

Hence f is two-to-one near 0, by the Local Mapping Theorem.

(b) Since $f'(2\pi i) = e^{2\pi i} = 1 \neq 0$, we deduce that f is one-to-one near $2\pi i$, by the corollary to the Local Mapping Theorem.

(c) The Taylor series about 0 for f is

$$f(z) = \sin z - z = -\frac{z^3}{3!} + \frac{z^5}{5!} - \cdots.$$

Hence f is three-to-one near 0, by the Local Mapping Theorem.

Solution to Exercise 3.9

(a) The Taylor series about $\alpha = 0$ for f is

$$f(z) = 3z + z^3.$$

Since f is an odd function, the Taylor series about $\beta = f(\alpha) = 0$ for f^{-1} is of the form

$$f^{-1}(w) = b_1 w + b_3 w^3 + b_5 w^5 + \cdots,$$

where $b_1, b_3, \ldots$ satisfy

$$z = b_1(3z + z^3) + b_3(3z + z^3)^3 + b_5(3z + z^3)^5 + \cdots.$$

Equating coefficients of powers of z, we obtain

$$\begin{aligned} z:\quad & 1 = 3b_1 && \Longrightarrow b_1 = \frac{1}{3}, \\ z^3:\quad & 0 = b_1 + 27b_3 && \Longrightarrow b_3 = -\frac{b_1}{27} = -\frac{1}{81}, \\ z^5:\quad & 0 = 27b_3 + 243b_5 && \Longrightarrow b_5 = -\frac{b_3}{9} = \frac{1}{729}. \end{aligned}$$

Hence

$$f^{-1}(w) = \tfrac{1}{3}w - \tfrac{1}{81}w^3 + \tfrac{1}{729}w^5 - \cdots.$$

(b) The Taylor series about $\alpha = 1$ for f is

$$\begin{aligned} f(z) = e^z &= ee^{z-1} \\ &= e\left(1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\right). \end{aligned}$$

Thus the Taylor series about $\beta = f(\alpha) = e$ for f^{-1} is of the form

$$f^{-1}(w) = b_0 + b_1(w-e) + b_2(w-e)^2 + \cdots,$$

where $b_0 = \alpha = 1$ and $b_1, b_2, \dots$ satisfy

$$\begin{aligned} z - 1 = b_1&\left(e(z-1) + \frac{e(z-1)^2}{2!} + \cdots\right) \\ &+ b_2\left(e(z-1) + \frac{e(z-1)^2}{2!} + \cdots\right)^2 + \cdots. \end{aligned}$$

Equating coefficients of powers of $z - 1$, we obtain

$$\begin{aligned} z-1:\quad & 1 = b_1 e && \Longrightarrow b_1 = \frac{1}{e}, \\ (z-1)^2:\quad & 0 = \frac{b_1 e}{2} + b_2 e^2 && \Longrightarrow b_2 = -\frac{b_1}{2e} = -\frac{1}{2e^2}. \end{aligned}$$

Hence

$$f^{-1}(w) = 1 + \frac{1}{e}(w-e) - \frac{1}{2e^2}(w-e)^2 + \cdots.$$

Solution to Exercise 4.1

(a) Since $f(z) = z^2 + 1$ is analytic and non-constant on $D = \{z : |z| < 1\}$ and continuous on $\overline{D} = \{z : |z| \le 1\}$, it follows from the Maximum Principle that there exists a point α in $\partial D = \{z : |z| = 1\}$ such that

$$\max\{|f(z)| : |z| \le 1\} = |f(\alpha)|.$$

Since each point of ∂D has the form e^{it}, for some $t \in [0, 2\pi)$, we need to determine

$$\max\{|f(e^{it})| : 0 \le t < 2\pi\}.$$

Because

$$f(e^{it}) = e^{2it} + 1 = e^{it}(e^{it} + e^{-it}) = 2e^{it}\cos t,$$

we obtain $|f(e^{it})|^2 = 4\cos^2 t$.

Now, the maximum of $4\cos^2 t$ on $[0, 2\pi)$ is 4, attained when $\cos t = \pm 1$ (that is, when $t = 0, \pi$, so $e^{it} = \pm 1$). Thus

$$\max\{|f(e^{it})| : 0 \le t < 2\pi\} = \sqrt{4} = 2,$$

so

$$\max\{|z^2 + 1| : |z| \le 1\} = 2.$$

By the Maximum Principle, this maximum can only be attained on ∂D, so it is attained at $z = \pm 1$ only.

Remark: This maximum could have been found by simply noting that, for $|z| \le 1$,

$$\begin{aligned} |z^2 + 1| &\le |z|^2 + 1 \\ &\le 1 + 1 = 2, \end{aligned}$$

by the Triangle Inequality, and that equality is obtained by taking $z = \pm 1$.

(b) Since $f(z) = z^2 - 2$ is analytic and non-constant on $D = \{z : |z - 1| < 1\}$ and continuous on $\overline{D} = \{z : |z - 1| \le 1\}$, it follows from the Maximum Principle that there exists a point α in $\partial D = \{z : |z - 1| = 1\}$ such that

$$\max\{|f(z)| : |z - 1| \le 1\} = |f(\alpha)|.$$

Since each point of ∂D has the form $1 + e^{it}$, for some $t \in [0, 2\pi)$, we need to determine

$$\max\{|f(1 + e^{it})| : 0 \le t < 2\pi\}.$$

Because

$$\begin{aligned} f(1 + e^{it}) &= (1 + e^{it})^2 - 2 \\ &= e^{2it} + 2e^{it} - 1 \\ &= e^{it}(e^{it} + 2 - e^{-it}) \\ &= 2e^{it}(i\sin t + 1), \end{aligned}$$

we obtain

$$|f(1 + e^{it})|^2 = 4(\sin^2 t + 1).$$

Now, the maximum of $4(\sin^2 t + 1)$ on $[0, 2\pi)$ is 8, attained when $\sin t = \pm 1$ (that is, when $t = \pi/2, 3\pi/2$, so $1 + e^{it} = 1 \pm i$). Thus

$$\max\{|f(1 + e^{it})| : 0 \le t < 2\pi\} = \sqrt{8} = 2\sqrt{2},$$

so

$$\max\{|z^2 - 2| : |z - 1| \le 1\} = 2\sqrt{2}.$$

By the Maximum Principle, this maximum can only be attained on ∂D, so it is attained at $z = 1 \pm i$ only.

(c) Since $f(z) = z^3 - 1$ is analytic and non-constant on the open square

$$S = \{z : 0 < \operatorname{Re} z < 1,\ 0 < \operatorname{Im} z < 1\}$$

and continuous on $\overline{S}$, it follows from the Maximum Principle that the required maximum is attained at some point of ∂S.

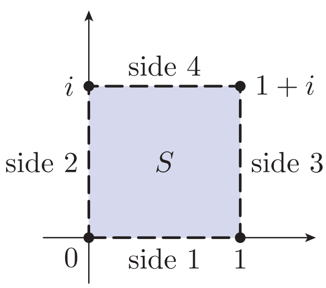
In order to study the four sides of the square S, we note that if $z = x + iy$, then

$$\begin{aligned} z^3 - 1 &= (x+iy)^3 - 1 \\ &= \left(x^3 - 3xy^2 - 1\right) + i\left(3x^2y - y^3\right), \end{aligned}$$

so

$$\begin{aligned} &|z^3 - 1|^2 \\ &= \left(x^3 - 3xy^2 - 1\right)^2 + \left(3x^2y - y^3\right)^2 \\ &= \left(x^6 + 9x^2y^4 + 1 - 6x^4y^2 + 6xy^2 - 2x^3\right) \\ &\quad + \left(9x^4y^2 + y^6 - 6x^2y^4\right) \\ &= x^6 + y^6 + 3x^4y^2 + 3x^2y^4 + 6xy^2 - 2x^3 + 1. \end{aligned}$$

There are four sides to consider (illustrated below).



Side 1: $y = 0$, $0 \le x \le 1$

In this case

$$|z^3 - 1|^2 = x^6 - 2x^3 + 1 = \left(x^3 - 1\right)^2,$$

which attains a maximum value (on $[0, 1]$) of 1 at $x = 0$.

Side 2: $x = 0$, $0 \le y \le 1$

In this case

$$|z^3 - 1|^2 = y^6 + 1,$$

which attains a maximum value (on $[0, 1]$) of 2 at $y = 1$.

Side 3: $x = 1$, $0 \le y \le 1$

In this case

$$|z^3 - 1|^2 = y^6 + 3y^4 + 9y^2,$$

which attains a maximum value (on $[0, 1]$) of 13 at $y = 1$.

Side 4: $y = 1$, $0 \le x \le 1$

In this case

$$|z^3 - 1|^2 = x^6 + 3x^4 - 2x^3 + 3x^2 + 6x + 2.$$

Now, the function

$$g(x) = x^6 + 3x^4 - 2x^3 + 3x^2 + 6x + 2$$

attains its maximum value of 13 on $[0, 1]$ at 1, since

$$\begin{aligned} g'(x) &= 6x^5 + 12x^3 - 6x^2 + 6x + 6 \\ &\ge -6x^2 + 6 \ge 0, \end{aligned}$$

for $0 \le x \le 1$. Thus

$$\max\{|f(z)| : z \in \partial S\} = \sqrt{13},$$

and hence

$\max\left\{|z^3 - 1| : 0 \le \operatorname{Re} z \le 1,\ 0 \le \operatorname{Im} z \le 1\right\} = \sqrt{13}.$

By the Maximum Principle, this maximum can only be attained on ∂S, so it is attained at $z = 1 + i$ only.

Solution to Exercise 4.2

Since $f(z) = \exp(z^2)$ is analytic and non-constant on $D = \{z : |z| < 1\}$, and continuous and non-zero on $\overline{D} = \{z : |z| \le 1\}$, it follows from the Minimum Principle that there exists $\alpha \in \partial D = \{z : |z| = 1\}$ such that

$$\min\left\{|\exp(z^2)| : |z| \le 1\right\} = |f(\alpha)|.$$

Since each point of ∂D has the form e^{it}, for $t \in [0, 2\pi)$, we need to determine

$$\min\left\{|\exp(e^{2it})| : 0 \le t < 2\pi\right\}.$$

Now, $|\exp(e^{2it})| = \exp(\cos 2t)$, so

$$\begin{aligned} &\min\left\{|\exp(e^{2it})| : 0 \le t < 2\pi\right\} \\ &= \min\{\exp(\cos 2t) : 0 \le t < 2\pi\} \\ &= e^{-1}, \end{aligned}$$

attained when $\cos 2t = -1$ (that is, when $t = \pi/2, 3\pi/2$, so $e^{it} = \pm i$). Thus

$$\min\left\{|\exp(z^2)| : |z| \le 1\right\} = e^{-1}.$$

By the Minimum Principle, this minimum can only be attained on ∂D, so it is attained at $z = \pm i$ only.

Solution to Exercise 4.3

The function $h = f - g$ is analytic on $\mathcal{R}$ and continuous on $\overline{\mathcal{R}}$. Since $f = g$ on $\partial\mathcal{R}$, we see that $h = 0$ on $\partial\mathcal{R}$.

Suppose, in order to reach a contradiction, that h is non-constant on $\mathcal{R}$. Then the Maximum Principle tells us that $|h|$ attains its maximum on $\overline{\mathcal{R}}$ at some point of $\partial\mathcal{R}$. However, $|h| = 0$ on $\partial\mathcal{R}$, so $|h| = 0$ on $\mathcal{R}$. Thus, contrary to our assumption, h is constant on $\mathcal{R}$ after all.

Hence h is constant on $\mathcal{R}$, and continuous on $\overline{\mathcal{R}}$, and it takes the value 0 on $\partial\mathcal{R}$. It follows that $h = 0$ throughout $\overline{\mathcal{R}}$, so $f = g$ on $\mathcal{R}$.

Solution to Exercise 4.4

Since $f(D) = D$, where $D = \{z : |z| < 1\}$, we see that

$$|f(z)| < 1, \quad \text{for } |z| < 1.$$

Since f is analytic on D with $f(0) = 0$, it follows from Schwarz's Lemma with $M = 1$ and $R = 1$ that

$$|f(z)| \leq |z|, \quad \text{for } |z| < 1. \tag{S2}$$

Since f is one-to-one and analytic on D, f^{-1} exists and is analytic on $f(D) = D$ (by the Inverse Function Rule). Also, $f^{-1}(0) = 0$ (since $f(0) = 0$). Hence, replacing f by f^{-1} in the argument leading to inequality (S2), we obtain

$$|f^{-1}(z)| \leq |z|, \quad \text{for } |z| < 1. \tag{S3}$$

Replacing z by $f(z)$ in inequality (S3) gives

$$|f^{-1}(f(z))| \leq |f(z)|, \quad \text{for } |f(z)| < 1,$$

that is,

$$|z| \leq |f(z)|, \quad \text{for } |z| < 1. \tag{S4}$$

From inequalities (S2) and (S4), $|f(z)| = |z|$, for $|z| < 1$, so the function

$$g(z) = \frac{f(z)}{z} \quad (z \in D - \{0\})$$

is analytic on $D - \{0\}$ and satisfies

$$|g(z)| = 1, \quad \text{for } z \in D - \{0\}. \tag{S5}$$

It follows from equation (S5) that g has a local maximum at each point of $D - \{0\}$.

Hence g is constant on $D - \{0\}$, that is,

$$g(z) = \lambda, \quad \text{for } z \in D - \{0\},$$

where $|\lambda| = 1$. Hence

$$f(z) = \lambda z, \quad \text{for } z \in D - \{0\},$$

and this equality also holds for $z = 0$, giving the desired result.

Solution to Exercise 4.5

(a) In this part there is no need to use the Maximum Principle. For $|z| \leq 1$, we have

$$|z^2 + 2| \leq |z|^2 + 2 \leq 3,$$

by the Triangle Inequality. If $|z| < 1$, then the above inequality is a strict inequality, so $|z^2 + 3|$ does not attain the value 3.

If $|z| = 1$, then we can write $z = e^{it}$ for some $t \in [0, 2\pi)$. Then

$$z^2 + 2 = e^{2it} + 2 = (\cos 2t + 2) + i \sin 2t.$$

Hence

$$\begin{aligned} |z^2 + 2|^2 &= (\cos 2t + 2)^2 + \sin^2 2t \\ &= \cos^2 2t + 4\cos 2t + 4 + \sin^2 2t \\ &= 4\cos 2t + 5. \end{aligned}$$

The maximum of this expression is 9, attained when $\cos 2t = 1$, that is, when $t = 0, \pi$, so $e^{it} = \pm 1$. We deduce that

$$\max\{|z^2 + 2| : |z| \leq 1\} = 3,$$

and the maximum is attained only when $z = \pm 1$.

(b) Since the function $f(z) = z^2 - 2$ is analytic and non-constant on the open disc $D = \{z : |z - i| < 1\}$, and continuous on $\overline{D} = \{z : |z - i| \leq 1\}$, it follows from the Maximum Principle that there exists $\alpha \in \partial D = \{z : |z - i| = 1\}$ such that

$$\max\{|f(z)| : |z - i| \leq 1\} = |f(\alpha)|.$$

Furthermore, the maximum value is not attained at any point inside D.

Since each point of ∂D has the form $i + e^{it}$, for some $t \in [0, 2\pi)$, we need to determine

$$\max\{|f(i + e^{it})| : 0 \leq t < 2\pi\}.$$

Observe that

$$\begin{aligned}f(i+e^{it}) &= (i+e^{it})^2 - 2\\ &= -3 + 2ie^{it} + e^{2it}\\ &= e^{it}(-3e^{-it} + 2i + e^{it})\\ &= e^{it}(-3(\cos t - i\sin t) + 2i\\ &\quad + (\cos t + i\sin t))\\ &= e^{it}(-2\cos t + 2i(2\sin t + 1)).\end{aligned}$$

Now, $|e^{it}| = 1$, so

$$\begin{aligned}|f(i+e^{it})|^2 &= 4\cos^2 t + 4(2\sin t + 1)^2\\ &= 4(1-\sin^2 t) + 4(4\sin^2 t + 4\sin t + 1)\\ &= 12\sin^2 t + 16\sin t + 8\\ &= 12\left(\sin^2 t + \tfrac{4}{3}\sin t + \tfrac{2}{3}\right)\\ &= 12\left(\left(\sin t + \tfrac{2}{3}\right)^2 + \tfrac{2}{9}\right).\end{aligned}$$

The maximum of this expression is 36, attained when $\sin t + \frac{2}{3} = \frac{5}{3}$, that is, when $\sin t = 1$, which corresponds to $t = \pi/2$ and $\alpha = i + e^{it} = 2i$. Thus

$$\max\left\{\left|f\left(i+e^{it}\right)\right| : 0 \le t < 2\pi\right\} = \sqrt{36} = 6.$$

We deduce that

$$\max\left\{|z^2 - 2| : |z - i| \le 1\right\} = 6,$$

and the maximum is attained only when $z = 2i$.

(c) In this part there is no need to use the Maximum Principle. For $z = x + iy$, we have

$$\left|e^{z^2}\right| = e^{\mathrm{Re}(z^2)} = e^{x^2-y^2}.$$

Now, if $-1 \le x \le 1$ and $-1 \le y \le 1$, then $x^2 - y^2 \le 1$, so

$$\left|e^{z^2}\right| \le e^1 = e.$$

Equality is attained when $x^2 - y^2 = 1$, that is, when $x^2 = 1$ and $y = 0$, which corresponds to $z = \pm 1$. We deduce that

$$\max\left\{\left|e^{z^2}\right| : -1 \le \mathrm{Re}\, z \le 1,\ -1 \le \mathrm{Im}\, z \le 1\right\} = e,$$

and the maximum is attained only when $z = \pm 1$.

(d) Since $f(z) = \tan z$ is analytic and non-constant on the open rectangle

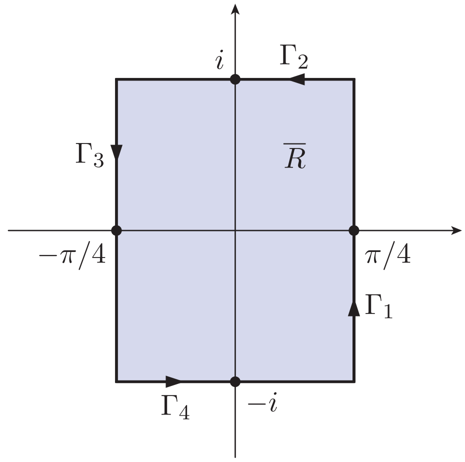
$$R = \{z : -\pi/4 < \mathrm{Re}\, z < \pi/4,\ -1 < \mathrm{Im}\, z < 1\},$$

and continuous on $\overline{R}$, it follows from the Maximum Principle that there exists $\alpha \in \partial R$ such that

$$\max\{|f(z)| : z \in \overline{R}\} = |f(\alpha)|.$$

Furthermore, the maximum value is not attained at any point inside R. Thus we need only find the maximum value of $|f(z)|$ for $z \in \partial R$.

Let $\partial R = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ as in the figure.



Since $f(z) = \tan z$ is an odd function, it follows that $|f(-z)| = |f(z)|$, for all $z \in \overline{\mathcal{R}}$. Therefore $|f(z)|$ takes the same values on Γ_3 as it does on Γ_1, and it takes the same values on Γ_4 as it does on Γ_2. Consequently, to find the maximum value of $|f(z)|$ on $\overline{\mathcal{R}}$, we need only consider the values of $|f(z)|$ on Γ_1 and Γ_2.

Using the hint, for $z \in \Gamma_1$, we have

$$|\tan z|^2 = \frac{\frac{1}{2} + \sinh^2 y}{\frac{1}{2} + \sinh^2 y} = 1, \quad \text{for } -1 \le y \le 1,$$

since $\sin \pi/4 = \cos \pi/4 = 1/\sqrt{2}$.

Now observe that $\tan^2 x \le 1$, for $-\pi/4 \le x \le \pi/4$, with strict inequality unless $x = \pm\pi/4$. Hence $\sin^2 x \le \cos^2 x$, for $-\pi/4 \le x \le \pi/4$, with strict inequality unless $x = \pm\pi/4$. It follows that, for $z \in \Gamma_2$, we have

$$|\tan z|^2 = \frac{\sin^2 x + \sinh^2 1}{\cos^2 x + \sinh^2 1} \le 1,$$

with strict inequality unless z is one of the endpoints $\pm\pi/4 + i$ of Γ_2.

In summary, we have

$$\max\{|\tan z| : z \in \overline{R}\} = \max\{|\tan z| : z \in \partial R\} = 1,$$

and the maximum is attained at all points on Γ_1 and Γ_3, and at no other points.

Solution to Exercise 4.6

(a) In the proof of Schwarz's Lemma we found that

$$\left|\frac{f(z)}{z}\right| \le \frac{M}{R}, \quad \text{for } 0 < |z| < R.$$

Since

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{f(z)}{z},$$

we deduce that

$$|f'(0)| = \lim_{z\to 0}\left|\frac{f(z)}{z}\right| \le \frac{M}{R},$$

as required.

(b) Define the analytic function g as in the proof of Schwarz's Lemma. If

$$\left|\frac{f(z_0)}{z_0}\right| = \frac{M}{r},$$

for some z_0 with $0 < |z_0| < R$, then the function g must have a local maximum at z_0, which is possible only if g is constant, say $g(z) = \lambda$ for $0 < |z| < R$. Hence $f(z) = \lambda z$, for $|z| < R$, where $|\lambda| = M/R$, as required.

If $|f'(0)| = M/R$, then $|g(0)| = |f'(0)| = M/R$, so the function g has a local maximum at 0, which is again possible only if g is constant. Thus, once again, $f(z) = \lambda z$, for $|z| < R$, where $|\lambda| = M/R$.

Solution to Exercise 5.1

Since $z^n \to 0$ as $n \to \infty$, for $|z| < 1$,

$$f_n(z) = \frac{1}{1+z^n} \to \frac{1}{1+0} = 1 \text{ as } n \to \infty,$$

for $|z| < 1$. Hence (f_n) converges pointwise to the function $f(z) = 1$ on $E = \{z : |z| \le r\}$, for each r with $0 < r < 1$.

Also, for $|z| \le r$,

$$\begin{aligned}
|f_n(z) - f(z)| &= \left|\frac{1}{1+z^n} - 1\right| \\
&= \left|\frac{-z^n}{1+z^n}\right| \\
&\le \frac{|z|^n}{1-|z|^n} \\
&\le \frac{r^n}{1-r^n},
\end{aligned}$$

where we have applied the backwards form of the Triangle Inequality to obtain the second-to-last line. Since $a_n = r^n/(1-r^n)$, $n = 1, 2, \ldots$, is a null sequence of positive terms, we deduce, by the strategy for proving uniform convergence, that (f_n) converges uniformly to the function $f(z) = 1$ on E.

Solution to Exercise 5.2

The power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ is of the form $\displaystyle\sum_{n=0}^{\infty} \phi_n(z)$, where

$$\phi_n(z) = a_n z^n, \quad \text{for } n = 0, 1, 2, \ldots.$$

Let $E = \{z : |z| \le r\}$, where $0 < r < R$. Then

$$|\phi_n(z)| = |a_n||z|^n \le |a_n|r^n, \quad \text{for } z \in E.$$

Hence assumption 1 of the M-test holds with $M_n = |a_n|r^n$, for $n = 0, 1, 2, \ldots$. We now use the hint to show that assumption 2 of the M-test holds. Since $r < R$, the power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent at r; that is,

$$\sum_{n=0}^{\infty} |a_n r^n| = \sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} M_n$$

converges. Hence assumption 2 holds.

Thus, by the M-test, the power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ is uniformly convergent on E.

Solution to Exercise 5.3

Here

$$\phi_n(z) = \frac{z^n}{n^2}, \quad \text{for } n = 1, 2, \ldots,$$

and

$$|\phi_n(z)| = \frac{|z|^n}{n^2} \le \frac{1}{n^2},$$

for $z \in E = \{z : |z| \le 1\}$. Hence assumption 1 of the M-test holds with $M_n = 1/n^2$, for $n = 1, 2, \ldots$. Since $\displaystyle\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, assumption 2 of the M-test also holds.

Thus, by the M-test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ is uniformly convergent on E.

Solution to Exercise 5.4

In Example 5.4 we proved that the zeta function ζ is analytic on $\mathcal{R} = \{z : \operatorname{Re} z > 1\}$, and that

$$\zeta'(z) = -\sum_{n=2}^{\infty} \frac{\log n}{n^z}, \quad \text{for } \operatorname{Re} z > 1. \tag{S6}$$

The formula for $\zeta''(z)$ can be obtained by term-by-term differentiation of equation (S6). (See Remark 3 after Weierstrass' Theorem and Remark 1 after Example 5.4.)

Since $n^{-z} = e^{-z\log n}$, we obtain

$$\begin{aligned}\zeta''(z) &= -\sum_{n=2}^{\infty} \frac{d}{dz}\left(\log n \; e^{-z\log n}\right)\\ &= -\sum_{n=2}^{\infty} -(\log n)^2 e^{-z\log n}\\ &= \sum_{n=2}^{\infty} \frac{(\log n)^2}{n^z}, \quad \text{for } \operatorname{Re} z > 1.\end{aligned}$$

Solution to Exercise 5.5

We use the strategy for proving uniform convergence. First note that, for any $z \in \mathbb{C}$,

$$f_n(z) = z + \frac{z^2}{n} \to z + 0 = z \text{ as } n \to \infty.$$

Hence (f_n) converges pointwise to the limit function $f(z) = z$ on $\mathbb{C}$, so it converges pointwise to f on any closed disc in $\mathbb{C}$.

Now let $E = \{z : |z| \le r\}$, for some $r > 0$. Then

$$\begin{aligned}|f_n(z) - f(z)| &= \left|z + \frac{z^2}{n} - z\right|\\ &= \frac{|z|^2}{n}\\ &\le \frac{r^2}{n},\end{aligned}$$

for $n = 1, 2, \dots$ and all $z \in E$. Since (r^2/n) is a null sequence for each fixed $r > 0$, we deduce that (f_n) converges uniformly to f on E.

Solution to Exercise 5.6

We apply the M-test. Here

$$\phi_n(z) = \frac{z^n}{1+z^n}, \quad n = 1, 2, \dots.$$

Let $E = \{z : |z| \le r\}$, where $0 < r < 1$. Then

$$|\phi_n(z)| \le \frac{|z|^n}{1-|z|^n} \le \frac{r^n}{1-r^n}, \quad \text{for } z \in E,$$

where we have used the backwards form of the Triangle Inequality and the fact that $|z|^n \le r^n$.

Hence assumption 1 of the M-test holds with

$$M_n = \frac{r^n}{1-r^n}, \quad \text{for } n = 1, 2, \dots.$$

Since

$$\frac{r^n}{1-r^n} \le \frac{r^n}{1-r}, \quad \text{for } n = 1, 2, \dots,$$

and

$$\frac{1}{1-r}\sum_{n=1}^{\infty} r^n$$

converges (because $0 < r < 1$), we deduce from the Comparison Test that $\sum_{n=1}^{\infty} M_n$ converges. Thus assumption 2 of the M-test also holds, so

$$\sum_{n=1}^{\infty} \frac{z^n}{1+z^n}$$

is uniformly convergent on each closed disc $E = \{z : |z| \le r\}$, for $0 < r < 1$.

Any closed disc in $\{z : |z| < 1\}$ – even one that is not centred at 0 – is contained in a set E, for some $r > 0$, so we see that the series is uniformly convergent on each closed disc in $\{z : |z| < 1\}$.

It follows from Weierstrass' Theorem that the sum function f is analytic on the open unit disc $\{z : |z| < 1\}$. Furthermore, we can obtain the derivative of f by term-by-term differentiation of the series:

$$\begin{aligned}f'(z) &= \sum_{n=1}^{\infty} \frac{d}{dz}\left(\frac{z^n}{1+z^n}\right)\\ &= \sum_{n=1}^{\infty} \frac{(1+z^n)nz^{n-1} - nz^{n-1} \times z^n}{(1+z^n)^2}\\ &= \sum_{n=1}^{\infty} \frac{nz^{n-1}}{(1+z^n)^2}, \quad \text{for } |z| < 1.\end{aligned}$$

Solution to Exercise 6.1

(a) We use the substitution

$$t = nu, \quad dt = n\,du, \quad \text{for } n = 1, 2, \ldots.$$

Then, since

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt, \quad \text{for } \operatorname{Re} z > 1,$$

and since $n > 0$ and $u > 0$, we have

$$\begin{aligned}\Gamma(z) &= \int_0^\infty e^{-nu}(nu)^{z-1} n\,du \\ &= n^z \int_0^\infty e^{-nu} u^{z-1}\,du,\end{aligned}$$

for $n = 1, 2, \ldots$ and $\operatorname{Re} z > 1$.

The result follows on replacing the integration variable u by t.

(b) We use the substitution

$$t = -\log u, \quad dt = -\frac{1}{u}\,du.$$

Then, since

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt, \quad \text{for } \operatorname{Re} z > 1,$$

we have

$$\begin{aligned}\Gamma(z) &= \int_1^0 e^{\log u}(-\log u)^{z-1}\left(-\frac{1}{u}\right) du \\ &= \int_0^1 u\left(\log\frac{1}{u}\right)^{z-1}\frac{1}{u}\,du \\ &= \int_0^1 \left(\log\frac{1}{u}\right)^{z-1} du, \quad \text{for } \operatorname{Re} z > 1.\end{aligned}$$

The result follows on replacing the integration variable u by t.

Solution to Exercise 6.2

(a) By the functional equation for the gamma function and Theorem 6.5, we have

$$\begin{aligned}\Gamma\left(\tfrac{5}{2}\right) &= \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) \\ &= \tfrac{3}{2} \times \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) \\ &= \tfrac{3}{2} \times \tfrac{1}{2}\sqrt{\pi} = \tfrac{3}{4}\sqrt{\pi}.\end{aligned}$$

(b) By the functional equation for the gamma function and Theorem 6.5, we have

$$\begin{aligned}\Gamma\left(-\tfrac{1}{2}\right) &= \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} \\ &= \frac{\sqrt{\pi}}{-\frac{1}{2}} = -2\sqrt{\pi}.\end{aligned}$$

Solution to Exercise 6.3

(a) By the given formula and the formula $\Gamma(\frac{3}{2}) = \frac{1}{2}\pi^{1/2}$ calculated in the proof of Theorem 6.5,

$$V_1 = \frac{\pi^{1/2}}{\Gamma\left(\frac{3}{2}\right)} r = \frac{\pi^{1/2}}{\frac{1}{2}\pi^{1/2}} r = 2r$$

(the length of the real interval $(-r, r)$).

(b) By the given formula,

$$V_2 = \frac{\pi}{\Gamma(2)} r^2 = \frac{\pi}{1} r^2 = \pi r^2$$

(the area of a disc of radius r).

(c) By the given formula and Exercise 6.2(a),

$$V_3 = \frac{\pi^{3/2}}{\Gamma\left(\frac{5}{2}\right)} r^3 = \frac{\pi^{3/2}}{\frac{3}{4}\pi^{1/2}} r^3 = \tfrac{4}{3}\pi r^3$$

(the volume of a three-dimensional ball of radius r).

(d) By the given formula,

$$V_4 = \frac{\pi^2}{\Gamma(3)} r^4 = \frac{\pi^2}{2} r^4 = \tfrac{1}{2}\pi^2 r^4.$$

Solution to Exercise 6.4

If $\alpha = \beta = \frac{1}{2}$, then the integral is

$$\int_0^1 t^{-1/2}(1-t)^{-1/2}\,dt = \int_0^1 \frac{1}{\sqrt{t - t^2}}\,dt.$$

Making the substitution $t = \sin^2\theta$, so $dt = 2\sin\theta\cos\theta\,d\theta$, we obtain

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{t-t^2}}\,dt &= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{\sqrt{\sin^2\theta - \sin^4\theta}}\,d\theta \\ &= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{\sqrt{\sin^2\theta(1-\sin^2\theta)}}\,d\theta \\ &= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{\sqrt{\sin^2\theta\cos^2\theta}}\,d\theta \\ &= \int_0^{\pi/2} 2\,d\theta = \pi.\end{aligned}$$

On the other hand, by Theorem 6.5,

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left(\sqrt{\pi}\right)^2}{1} = \pi,$$

so the identity is verified for $\alpha = \beta = \frac{1}{2}$.

Unit C3

Conformal mappings

Introduction

In this unit we continue the discussion of conformal functions that we began in Unit A4. There we described a function as *conformal* if it preserves angles between smooth paths. We learned from Theorem 4.2 of Unit A4 that any function that is analytic on a region, with a derivative that is never zero, is conformal on that region.

Much of this unit is concerned with developing techniques for finding the images of regions under various conformal functions, or *conformal mappings* as they are often called. We will also look at techniques for finding specific one-to-one conformal mappings that map one given region onto another. Such techniques will prove useful in Unit D1 when we study fluid flow around an obstacle. Observe that not all conformal mappings are one-to-one; for instance, the function

$$f(z) = e^z \quad (z \in \mathbb{C})$$

is conformal, but it is not one-to-one because, for example, $f(0) = f(2\pi i)$.

Many of the regions we consider have boundaries that consist of line segments and arcs of circles. By understanding how lines and circles behave under various conformal mappings, we can sometimes see how to map the boundary of one such region onto another. Having dealt with the boundary, we can then deal with the region it encloses.

In Section 1 we show that linear functions and the reciprocal function 'preserve' lines and circles; that is, they map any line or circle to a line or circle. We also introduce the *point at infinity* and the *extended complex plane*, which make it possible to think of a line extended to infinity as a *generalised circle*.

In Section 2 we consider the class of all functions that can be obtained by composing linear functions and the reciprocal function. These functions are called *Möbius transformations*. We analyse the properties of these transformations and show that any line or circle can be mapped to any other line or circle by a suitable Möbius transformation.

In Section 3 we develop techniques for finding the images of lines and circles under a Möbius transformation. One of these techniques leads to a different type of equation for a circle, known as the *Apollonian form*. To help us analyse this form, we introduce the notion of *inverse points* with respect to a circle.

In Section 4 we use the techniques from Section 3 to find Möbius transformations that map one disc or half-plane onto another. We then explore ways of constructing conformal mappings between more general regions by composing Möbius transformations with some of the other conformal mappings that have been introduced in the module.

Finally, in Section 5 we discuss a remarkable theorem due to Riemann, which states that there exists a one-to-one conformal mapping from any simply connected region other than $\mathbb{C}$ onto the open unit disc.

Unit guide

The material in the first three sections paves the way for the work on regions in Section 4, which is the most important section of the unit.

Section 5 is intended for reading only.

1 Linear and reciprocal functions

After working through this section, you should be able to:

- write down a linear function that maps one given region onto another region of the same shape
- find the image of a line or circle under the *reciprocal function*
- understand the definition of a rational function on the *extended complex plane*
- visualise the extended complex plane as the *Riemann sphere*.

1.1 Linear functions

Perhaps the simplest conformal mappings are those that map regions of the complex plane onto one another without distortion. For example, consider the function $f(z) = (1+i)z$. In Subsection 3.2 of Unit A2 we investigated the behaviour of f by examining its effect on a Cartesian grid. We showed that f scales the grid by the factor $|1+i| = \sqrt{2}$ and rotates it about the origin through the angle $\operatorname{Arg}(1+i) = \pi/4$, as shown in Figure 1.1. Lines map to lines, and angles are preserved (because the function is conformal), so the shaded square region remains square.

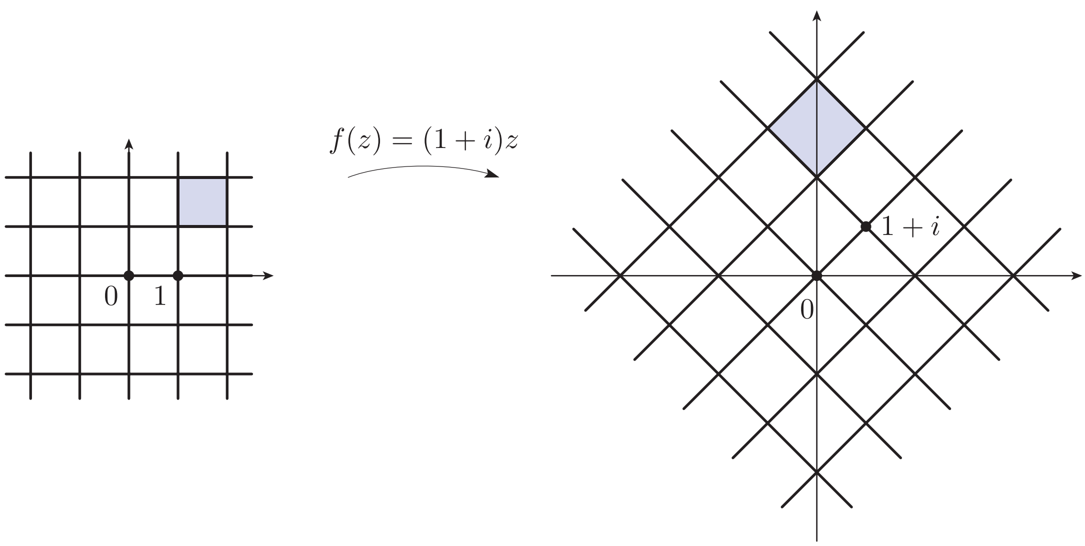


Figure 1.1 Image of a Cartesian grid under the linear function $f(z) = (1+i)z$

Scalings and rotations are not the only transformations of the complex plane that leave shapes of regions unaltered. Translations, which are functions of the form $f(z) = z + b$, for $b \in \mathbb{C}$, also preserve shapes (see Figure 1.2), and they are certainly conformal mappings too.

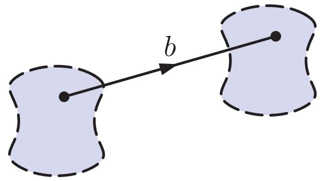


Figure 1.2 A shape preserved under a translation

Note that reflections, which map shapes to their mirror images, have the property that they *reverse* the signs of angles, so they are not conformal mappings.

In general, any conformal mapping that preserves shapes can be expressed as a composition of a scaling, followed by a rotation, followed by a translation; it can therefore be written in the form $f(z) = az + b$, where the coefficients a and b are complex numbers and $a \neq 0$. Functions of this form are called *linear functions*, because they map lines onto lines.

Definition

A function of the form $f(z) = az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$, is called a **linear function**.

Note that linear functions are *not* the same as the 'linear maps' that are studied in the subject of linear algebra. In particular, a linear function does not necessarily map 0 to 0.

Given any two regions $\mathcal{R}$ and $\mathcal{S}$ in the complex plane that have the same shape, it is straightforward to construct a linear function that sends $\mathcal{R}$ onto $\mathcal{S}$. We first scale $\mathcal{R}$ by a positive factor r to obtain a region of the same size as $\mathcal{S}$. Next we rotate through an angle θ about the origin, to obtain a region that is aligned with $\mathcal{S}$. Finally, we superimpose the region on $\mathcal{S}$ by a translation by c units parallel to the real axis and d units parallel to the imaginary axis. The required function is then

$$f(z) = \left(re^{i\theta}\right)z + (c + id),$$

which is a linear function. This process is illustrated in Figure 1.3.

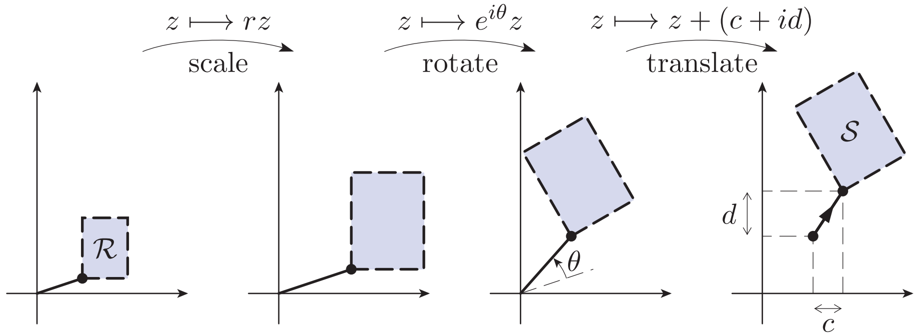


Figure 1.3 Composing a scaling, a rotation and a translation

Example 1.1

Find a linear function that maps the region $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ onto the region $\mathbb{C} - \{iy : y \in \mathbb{R},\ y \geq 2\}$. The regions are illustrated in Figure 1.4.

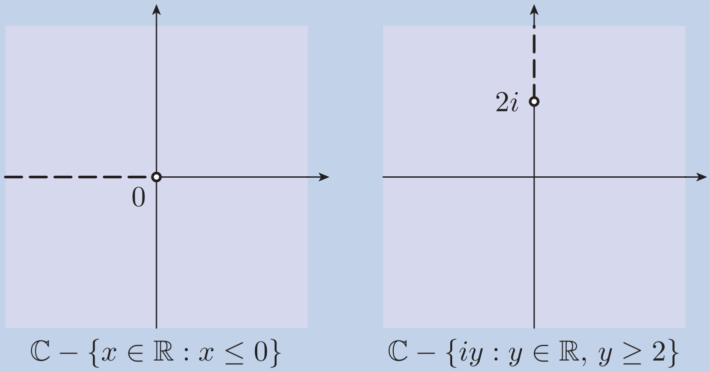


Figure 1.4 Two cut planes

Solution

Both regions have the same (unbounded) shape. We can map the first region onto the second region by a rotation through $-\pi/2$ about the origin followed by a translation of 2 units in the direction of the positive imaginary axis. No scaling is necessary. A suitable linear function is therefore

$$f(z) = e^{-i\pi/2}z + 2i = -iz + 2i.$$

Notice that there may be more than one linear function that maps one region onto another. Indeed, in Example 1.1 we could have scaled the region on the left by any positive factor r before carrying out the rotation and the translation, to give the linear function $f(z) = -irz + 2i$.

Exercise 1.1

Sketch the discs $D_1 = \{z : |z| < 2\}$ and $D_2 = \{z : |z - (1 + i)| < 4\}$, and find a linear function that maps D_1 onto D_2.

As we mentioned in the Introduction, one of the main objectives of this unit is to find conformal mappings between regions. In Exercise 1.1 you were asked to find a linear function that maps D_1 onto D_2. Notice, however, that to find such a function it would have been sufficient to find a linear function that maps the circular boundary of D_1 onto the circular boundary of D_2, because this linear function necessarily maps D_1 onto D_2.

For linear functions this observation about boundaries is rather obvious, but for other conformal mappings it is often easier to consider what

happens to the boundary of a region before dealing with the region itself. The boundaries of most of the regions we consider in Section 4 are made up of segments of lines and circles, so we will spend the rest of Section 1 and most of Sections 2 and 3 discussing lines and circles.

We begin by noting that the technique illustrated in Figure 1.3 gives the following result when applied to pairs of lines and pairs of circles.

Theorem 1.1

Linear functions map lines onto lines and circles onto circles. Furthermore:

(a) given any two lines L_1 and L_2, there is a linear function that maps L_1 onto L_2

(b) given any two circles C_1 and C_2, there is a linear function that maps C_1 onto C_2.

Now suppose that we wish to find a conformal mapping that maps a line onto a circle. The function cannot be linear, since a circle is not the same shape as a line. Instead we need a new type of function, which we introduce in the next subsection.

1.2 The reciprocal function

The **reciprocal function** is the function

$$f(z) = \frac{1}{z}.$$

Its domain and image set are $\mathbb{C} - \{0\}$. In Subsection 3.2 of Unit A2 we investigated the behaviour of this function by examining its effect on Cartesian and polar grids. The effect that f has on a Cartesian grid is shown in Figure 1.5 (with different scales for the left and right planes).

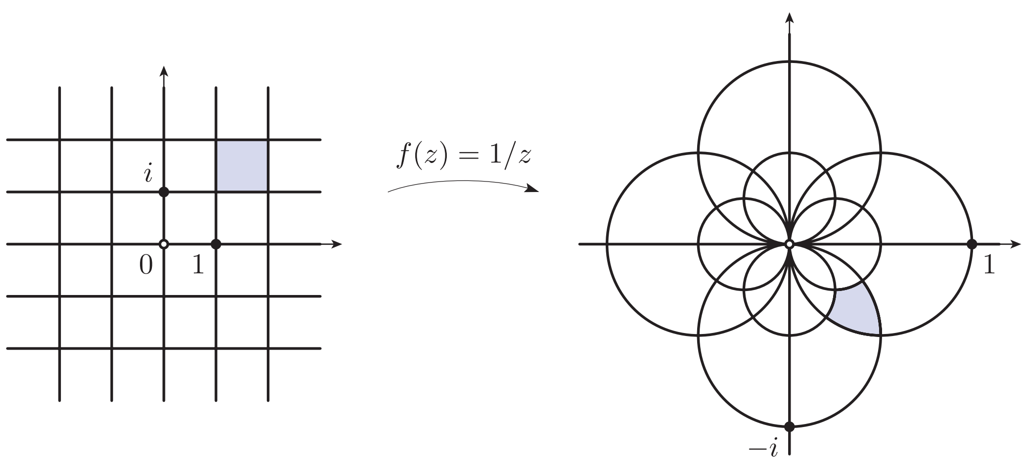


Figure 1.5 Image of a Cartesian grid under the reciprocal function

Figure 1.5 suggests that the reciprocal function might be the kind of function that we need in order to map lines onto circles. Indeed, apart from the lines that pass through the origin, each grid line is mapped to a circle. The next example shows what happens to a line that is not parallel to an axis.

Example 1.2

Find an equation for the image of the line $3y - 4x = 2$ under the reciprocal function $f(z) = 1/z$.

Solution

Let $w = u + iv$ be the image of an arbitrary point $z = x + iy$ on the line $3y - 4x = 2$. Since $w = f(z) = 1/z$, it follows that $z = 1/w$ and hence that

$$x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}.$$

Equating real parts and imaginary parts, we find that

$$x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}.$$

Since x and y satisfy the equation $3y - 4x = 2$, it follows that u and v must satisfy the equation

$$3\left(\frac{-v}{u^2 + v^2}\right) - 4\left(\frac{u}{u^2 + v^2}\right) = 2.$$

Multiplying both sides of this equation by $u^2 + v^2$ gives the simpler equation $-3v - 4u = 2(u^2 + v^2)$, which we can rearrange to obtain

$$u^2 + 2u + v^2 + \tfrac{3}{2}v = 0.$$

By completing the squares, we see that

$$(u + 1)^2 + \left(v + \tfrac{3}{4}\right)^2 = \tfrac{25}{16}.$$

This is the equation of the circle centred at $-1 - \frac{3}{4}i$ of radius $\frac{5}{4}$, illustrated in Figure 1.6.

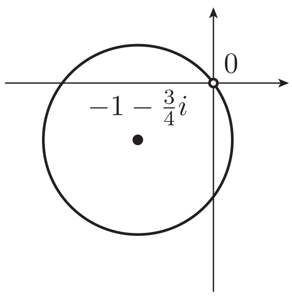


Figure 1.6 Circle centred at $-1 - \frac{3}{4}i$ of radius $\frac{5}{4}$

We have shown that every point in the image of the line $3y - 4x = 2$ under the reciprocal function $f(z) = 1/z$ satisfies this equation for a circle, but we have not shown that every point on the circle belongs to the image of the line $3y - 4x = 2$ under f. In fact, the point 0 lies on the circle, but it does not belong to the image of the line $3y - 4x = 2$ under f because 0 does not lie in the image set of f.

We will see later that the image of the line $3y - 4x = 2$ under the reciprocal function is the whole of the circle except for 0.

There is nothing special about the line $3y - 4x = 2$ in this example. We can use the same method to find equations for the images of other paths. We summarise the method in the form of a strategy. You will notice that this is different from the strategy for determining an image path used in Subsection 2.2 of Unit A2, and it applies only to the reciprocal function.

Strategy for finding an equation for the image of a path under the reciprocal function

To find an equation for the image $f(\Gamma)$ of a path Γ under $f(z) = 1/z$, apply the following steps.

1. Write down an equation that relates the x- and y-coordinates of all points $x + iy$ on Γ.
2. Replace x by $\dfrac{u}{u^2 + v^2}$ and y by $\dfrac{-v}{u^2 + v^2}$.
3. Simplify the resulting equation to obtain an equation that relates the u- and v-coordinates of all points $u + iv$ on the image $f(\Gamma)$.

As we saw in Example 1.2, the strategy gives an equation that is satisfied by all the points on the image $f(\Gamma)$. However, the equation may also be satisfied by some extra points that do not lie on the image (for example, the origin in Figure 1.6). We show how to deal with such missing points later in this section.

Exercise 1.2

Find an equation for the image of the line $y + 4x = 1$ under the reciprocal function.

The origin does not lie in the domain of the reciprocal function, so if a line (or circle) passes through the origin, then we must, for now, be content to find an equation for the image of what remains of the line (or circle) after the origin has been removed. We indicate one way to deal with the image of the origin later in this section.

Exercise 1.3

Find an equation for the image of the line $y - x = 0$ (with the origin removed) under the reciprocal function.

So far we have used the strategy to find equations for the images of lines under the reciprocal function. Some lines map to lines, whereas others map to circles. We now turn our attention to the images of circles.

Example 1.3

Find an equation for the image of the circle $(x-1)^2 + (y-2)^2 = 9$ under the reciprocal function.

Solution

This circle does not pass through the origin. It has equation

$$x^2 + y^2 - 2x - 4y - 4 = 0.$$

On replacing x by $u/\left(u^2+v^2\right)$ and y by $-v/\left(u^2+v^2\right)$, we obtain

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 2\left(\frac{u}{u^2+v^2}\right) - 4\left(\frac{-v}{u^2+v^2}\right) - 4 = 0.$$

The first two terms combine to give

$$\frac{u^2+(-v)^2}{(u^2+v^2)^2} = \frac{1}{u^2+v^2},$$

so the equation becomes

$$\left(\frac{1}{u^2+v^2}\right) - 2\left(\frac{u}{u^2+v^2}\right) - 4\left(\frac{-v}{u^2+v^2}\right) - 4 = 0.$$

On multiplying this equation by u^2+v^2, we obtain

$$1 - 2u + 4v - 4\left(u^2+v^2\right) = 0;$$

that is,

$$u^2 + \tfrac{1}{2}u + v^2 - v - \tfrac{1}{4} = 0.$$

By completing the squares, we obtain

$$\left(u+\tfrac{1}{4}\right)^2 + \left(v-\tfrac{1}{2}\right)^2 = \tfrac{9}{16},$$

which is an equation for the circle centred at $-\frac{1}{4}+\frac{1}{2}i$ of radius $\frac{3}{4}$.

Exercise 1.4

Find an equation for the image of each of the following circles under the reciprocal function. (Note that in part (b) the origin must be removed.)

(a) $x^2+y^2=4$ (b) $(x-3)^2+(y-4)^2=25$

It appears from the examples and exercises above that the reciprocal function maps any line or circle to a line or circle, with 0 omitted if necessary. To prove this in general, we need the following result, which shows that the equations of lines and circles can be expressed in a common algebraic form.

Theorem 1.2

Every line or circle has an equation of the form

$$a(x^2+y^2)+bx+cy+d=0,$$

where $a,b,c,d\in\mathbb{R}$ and $b^2+c^2>4ad$.

Conversely, any such equation represents a line or circle. Also:

(a) the equation represents a line if and only if $a=0$

(b) the line or circle passes through the origin if and only if $d=0$.

Proof First we prove that every line or circle has an equation of the form

$$a(x^2+y^2)+bx+cy+d=0, \tag{1.1}$$

where $a,b,c,d\in\mathbb{R}$ and $b^2+c^2>4ad$.

This is certainly true of lines, because each line is given by an equation of the form $px+qy=r$, where $p,q,r\in\mathbb{R}$ and p and q are not both 0. This has the required form with $a=0$, $b=p$, $c=q$ and $d=-r$, because $b^2+c^2=p^2+q^2$, which is greater than $4ad=0$.

It is true of circles too, because the circle centred at $p+iq$ of radius r is given by the equation $(x-p)^2+(y-q)^2=r^2$. Expanding the brackets, we obtain

$$x^2+y^2-2px-2qy+p^2+q^2-r^2=0.$$

This has the required form with $a=1$, $b=-2p$, $c=-2q$ and $d=p^2+q^2-r^2$, because

$$b^2+c^2=4(p^2+q^2)>4(p^2+q^2-r^2)=4ad.$$

Conversely, we now prove that (1.1) is the equation of a line or circle.

If $a=0$, then equation (1.1) reduces to $bx+cy=-d$, which is the equation of a line (b and c are not both zero because $b^2+c^2>0$).

If $a\neq 0$, then we can divide (1.1) through by a and complete the squares to obtain the following equation, which is equivalent to equation (1.1):

$$\left(x+\frac{b}{2a}\right)^2+\left(y+\frac{c}{2a}\right)^2=\frac{b^2+c^2-4ad}{4a^2}.$$

Since $b^2+c^2>4ad$, the right-hand side is positive, so this is the equation of a circle centred at $-(b+ic)/(2a)$ of radius $\sqrt{(b^2+c^2-4ad)/(4a^2)}$.

It remains to prove parts (a) and (b) of the theorem. In fact, part (a) has been established already, because we saw that if $a=0$, then equation (1.1) represents a line, and if $a\neq 0$, then it represents a circle.

For part (b), observe that the line or circle passes through the origin if and only if $x=y=0$ is a solution of equation (1.1), and this happens if and only if $d=0$. ■

As promised, we now prove that the reciprocal function maps any line or circle to a line or circle.

Theorem 1.3

The reciprocal function maps the set of non-zero points on the line or circle

$$a\left(x^2+y^2\right)+bx+cy+d=0,$$

where $a, b, c, d \in \mathbb{R}$ and $b^2+c^2>4ad$, onto the set of non-zero points on the line or circle

$$d\left(u^2+v^2\right)+bu-cv+a=0,$$

where $a, b, c, d \in \mathbb{R}$ and $b^2+(-c)^2>4da$.

Proof On replacing x by $u/\left(u^2+v^2\right)$ and y by $-v/\left(u^2+v^2\right)$ in the equation

$$a\left(x^2+y^2\right)+bx+cy+d=0,$$

for $x+iy\neq 0$, we obtain

$$a\left(\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2\right)+b\left(\frac{u}{u^2+v^2}\right)+c\left(\frac{-v}{u^2+v^2}\right)+d=0,$$

that is,

$$a\left(\frac{1}{u^2+v^2}\right)+b\left(\frac{u}{u^2+v^2}\right)+c\left(\frac{-v}{u^2+v^2}\right)+d=0,$$

where $u+iv\neq 0$. By multiplying through by u^2+v^2 and interchanging the first and last terms on the left-hand side of the equation, we obtain

$$d\left(u^2+v^2\right)+bu-cv+a=0.$$

This is an equation for a line or circle, because $a, b, c, d \in \mathbb{R}$ and

$$b^2+(-c)^2=b^2+c^2>4ad=4da.$$

Since the reciprocal function is its own inverse function, we can reverse this argument to see that every non-zero point on the image circle is the image of some point on the original circle. ■

Exercise 1.5

Prove that the reciprocal function maps:

(a) a line through the origin to a line through the origin

(b) a line not through the origin to a circle through the origin

(c) a circle through the origin to a line not through the origin

(d) a circle not through the origin to a circle not through the origin.

In each case, do not include the origin in a line or circle through the origin.

Note that (c) is the reverse of (b), which reflects the fact that the reciprocal function is its own inverse function.

The results of Exercise 1.5 are illustrated in Figure 1.7.

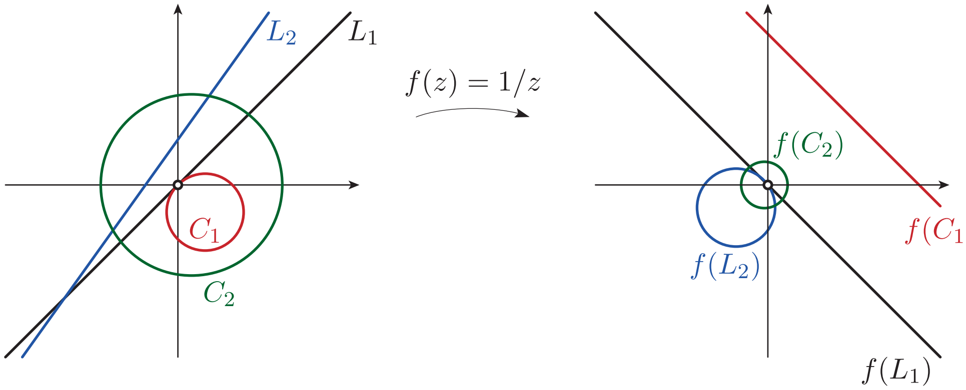


Figure 1.7 Images of lines and circles under the reciprocal function

There is no need to remember the results of Exercise 1.5 because you can work them out each time you need them, using the following intuitive argument.

First, on a line or circle that passes through the origin, we can choose points that are arbitrarily close to the origin. It follows that there are points on the image that are arbitrarily far from the origin and must therefore lie on a line. Second, on a line we can choose points that are arbitrarily far from the origin. It follows that there are points on the image that are arbitrarily close to the origin and must therefore lie on a line or circle through the origin. With a little practice, it is easy to use these two observations to predict the form taken by the image of any line or circle under the reciprocal function.

Exercise 1.6

Use Theorem 1.2 and Exercise 1.5 to determine whether the image under the reciprocal function of each of the following lines and circles is a line or a circle, and determine whether or not it passes through the origin.

(a) $2x + 3y = 6$ (b) $x^2 + (y-1)^2 = 1$ (c) $x^2 + (y-1)^2 = 2$

(d) $2x - 3y = 0$

1.3 The point at infinity

The discussion from the preceding subsection about the reciprocal function was rather untidy because of the differing properties of lines and circles, and because of the various exceptional cases involving the origin. We can deal with both difficulties in an elegant way by adjoining an additional point to the complex plane.

To see how this works, consider the correspondence between the circle and line that was investigated in Exercise 1.4(b), illustrated in Figure 1.8.

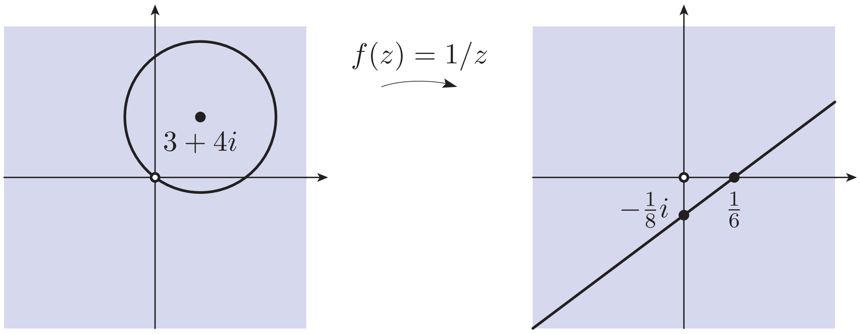


Figure 1.8 Image of the circle $(x-3)^2 + (y-4)^2 = 25$ under the reciprocal function

The circle in the z-plane on the left of Figure 1.8 passes through the origin, which is not in the domain of the function $f(z) = 1/z$. In Subsection 1.3 of Unit B4 we saw that if a function f has a removable singularity at a point α, then we can 'remove the singularity' at α by defining

$$f(\alpha) = \lim_{z \to \alpha} f(z).$$

The trouble with using this approach to 'remove the singularity' of the reciprocal function $f(z) = 1/z$ at 0 is that f has a pole at 0, so $f(z) \to \infty$ as $z \to 0$. It is therefore meaningless to define

$$f(0) = \lim_{z \to 0} f(z),$$

since the limit does not exist. One way to overcome this limitation is to consider ∞ to be a point, which we adjoin to the complex plane.

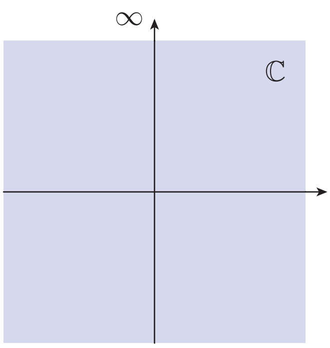


Figure 1.9 The complex plane and the point ∞

Definitions

The **extended complex plane** $\widehat{\mathbb{C}}$ (read 'C-hat') is the union of the ordinary complex plane $\mathbb{C}$ and one extra element, which is called the **point at infinity**, denoted by ∞. Thus $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

For the moment, you can think of ∞ as being an extra point that lies just 'beyond' the complex plane, as illustrated in Figure 1.9. In the next subsection we will see an intuitive way to picture ∞.

Let us return to considering the reciprocal function $f(z) = 1/z$. Working in the extended complex plane, we can define $f(0) = \infty$, since f has a pole at 0. More generally, if a function f has a pole at α, then $f(z) \to \infty$ as $z \to \alpha$, and we can extend the definition of f to α by defining $f(\alpha) = \infty$.

It is natural now to ask whether we can extend the definition of the reciprocal function to the whole of $\widehat{\mathbb{C}}$, by defining its value at ∞. We describe a manner for doing so that works for any rational function $f(z) = p(z)/q(z)$ (where p and q are polynomial functions).

Given a rational function f, and a point $\beta \in \widehat{\mathbb{C}}$, we write

$$f(z) \to \beta \text{ as } z \to \infty$$

to mean that

$$f(1/w) \to \beta \text{ as } w \to 0.$$

This makes sense, because if $z = 1/w$, then $z \to \infty$ as $w \to 0$. If $f(z) \to \beta$ as $z \to \infty$, then we can extend the domain of f to include the point ∞ by defining $f(\infty) = \beta$.

For example, if $f(z) = 1/z$, then

$$f(z) = \frac{1}{z} \to 0 \text{ as } z \to \infty,$$

so we define $f(\infty) = 0$. You will see more examples shortly, in Example 1.4 and Exercise 1.7.

Using these definitions, it can be shown that the domain of any rational function f can be extended to the whole of $\widehat{\mathbb{C}}$.

Strictly speaking, the function that we obtain by extending f in this way is *different* from f, because the two functions have different domains. However, we will write f for both functions, which should not cause confusion because it will be clear from the context which function we are dealing with. (Later on we will make the assumption that all such functions are in the extended form.)

Example 1.4

Determine how to extend the function

$$f(z) = \frac{4z+5}{2z-3}$$

to a function from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$.

Solution

The function f has a single singularity in $\mathbb{C}$, at $3/2$. Since

$$f(z) \to \infty \text{ as } z \to 3/2,$$

we define $f(3/2) = \infty$. Next,

$$f(z) = \frac{4z+5}{2z-3} = \frac{4+5/z}{2-3/z}.$$

Therefore

$$f(z) \to \frac{4+0}{2-0} = 2 \text{ as } z \to \infty,$$

so we define $f(\infty) = 2$. Hence the extended function is

$$f(z) = \begin{cases} \dfrac{4z+5}{2z-3}, & z \in \mathbb{C} - \{3/2\}, \\ \infty, & z = 3/2, \\ 2, & z = \infty. \end{cases}$$

It is important to observe that although ∞ is a member of the extended complex plane $\widehat{\mathbb{C}}$, along with all the complex numbers, we do not attempt to perform arithmetic operations with ∞ as if it were a complex number. We should treat it just as a point in $\widehat{\mathbb{C}}$, and not try to add with it, multiply with it, and so on.

Exercise 1.7

Determine how to extend the following functions to functions from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$.

(a) $f(z) = \dfrac{6z+4}{3z+1}$ (b) $f(z) = \dfrac{1}{z}$ (c) $f(z) = 5z + 7$

(d) $f(z) = \dfrac{1}{2z+1}$ (e) $f(z) = \dfrac{z^2+1}{z}$

After extending the domain of the reciprocal function to $\widehat{\mathbb{C}}$, the exceptional cases involving the origin, and the distinction between lines and circles, disappear. To see this, we look again at the correspondence between the circle and line in Exercise 1.4(b) (see Figures 1.8 and 1.10).

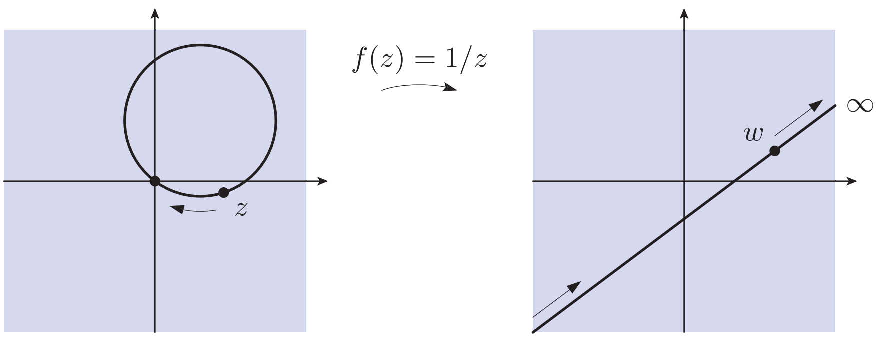


Figure 1.10 Image of the circle $(x-3)^2 + (y-4)^2 = 25$ under the reciprocal function

Since the point 0 maps to ∞ under the reciprocal function, we see that the image of the circle is a line together with the point ∞, so we need to consider how such a set can be interpreted.

To do this, think of a point z on the circle in Figure 1.10 moving clockwise towards the origin. As it moves, its image $w = f(z) = 1/z$ moves up the line. The closer that z gets to the origin, the further w moves up the line. When z reaches the origin, you can think of its image $w = f(z)$ reaching ∞. After that, as z continues clockwise round the circle, w returns up the line from below. If you like, you can think of the point ∞ as linking together the two ends of the line, thereby enabling points to travel 'round and round' the line.

There is nothing special about the line in Figure 1.10. Indeed, we can extend any line L in a similar way by forming the set $L \cup \{\infty\}$. Such a set is called an **extended line**.

Circles and extended lines will prove to be closely related in the work that follows, so it is helpful to make the following definition.

Definition

A **generalised circle** is a circle or an extended line.

Using this concept, we have the following result about linear functions and the reciprocal function, which treats these functions in their extended forms as functions from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$.

Theorem 1.4

Linear functions and the reciprocal function have the following properties:

(a) they are one-to-one mappings from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$

(b) they map generalised circles onto generalised circles.

Proof Suppose first that f is a linear function, given by $f(z) = az + b$, where $a \neq 0$. We already know that f is a one-to-one mapping from $\mathbb{C}$ onto $\mathbb{C}$, and it maps lines onto lines and circles onto circles.

Next, since $a \neq 0$, we see that $f(z) \to \infty$ as $z \to \infty$, so we define $f(\infty) = \infty$. Thus f is a one-to-one mapping from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$, and it maps extended lines onto extended lines.

Now consider the reciprocal function $f(z) = 1/z$, which maps $\mathbb{C} - \{0\}$ one-to-one onto $\mathbb{C} - \{0\}$. If C_1 is a generalised circle, then we know from Theorem 1.3 that f maps $C_1 - \{0, \infty\}$ onto $C_2 - \{0, \infty\}$, where C_2 is another generalised circle. Since $f(0) = \infty$ and $f(\infty) = 0$, it follows that f is a one-to-one mapping from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$, and it maps C_1 onto C_2. ■

1.4 The Riemann sphere

In this subsection we discuss a geometric model of the extended complex plane in which the point at ∞ appears as a genuine point and generalised circles become actual circles.

To describe this model, imagine that the complex plane $\mathbb{C}$ is embedded in three-dimensional space in such a way that each complex number $x + iy$ is represented by the point $(x, y, 0)$ in the (x, y)-plane. Now consider the sphere $\mathbb{S}$ of radius 1 centred at the origin, shown in Figure 1.11. This sphere is called the **Riemann sphere**, named after Bernhard Riemann, who studied the role of the Riemann sphere in complex analysis. (You have encountered Riemann several times before, in Books A and B, and in Unit C2.) By analogy with the Earth we will refer to the point $N = (0, 0, 1)$ at the top of the sphere as the *North Pole*.

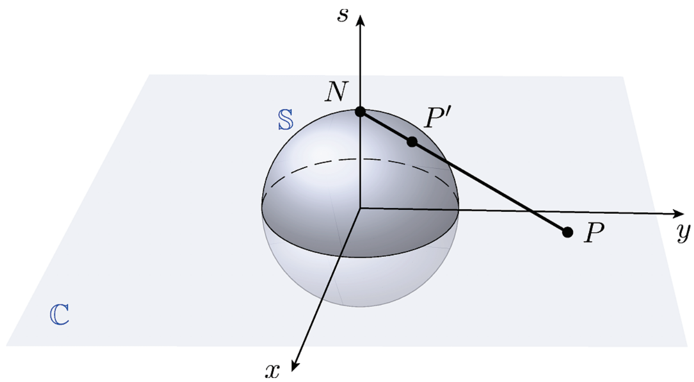


Figure 1.11 The Riemann sphere $\mathbb{S}$

Each line that joins a point P in the complex plane to the North Pole intersects the Riemann sphere at a point P', say, and vice versa. If P lies outside the unit circle $\{z : |z| = 1\}$ in $\mathbb{C}$, then P' lies in the upper half of $\mathbb{S}$, and if P lies inside the unit circle, then P' lies in the lower half of $\mathbb{S}$. And if P lies on the unit circle itself, where $\mathbb{S}$ intersects $\mathbb{C}$, then $P' = P$.

In this way we can associate all but one of the points P' on the Riemann sphere with the corresponding points P in the complex plane. The only point on the sphere that cannot be associated with a point in the complex plane is the North Pole. Of course, P' can be moved arbitrarily close to the North Pole by choosing P sufficiently far from the origin, but P' never actually reaches the North Pole. To fill this gap, we associate the North Pole with the point ∞ in the extended complex plane $\widehat{\mathbb{C}}$.

The function $\pi\colon \mathbb{S} \longrightarrow \widehat{\mathbb{C}}$ that projects the point P' on the Riemann sphere to the associated point P in the complex plane, and maps N to ∞, is called **stereographic projection**. Since this function is one-to-one and onto, we can use the Riemann sphere $\mathbb{S}$ as an alternative model of the extended complex plane $\widehat{\mathbb{C}}$.

One application to which the Riemann sphere is well suited is to help visualise generalised circles. To see this, consider a point P on a line L in $\mathbb{C}$ (see Figure 1.12).

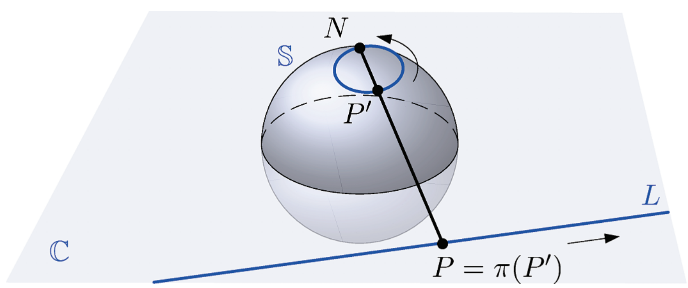


Figure 1.12 Correspondence between an extended line in $\widehat{\mathbb{C}}$ and a circle through the North Pole on the Riemann sphere

As P moves along L, the line that joins P to the North Pole N sweeps out a plane that intersects the Riemann sphere in a circle. So as P moves along L, the corresponding point P' on the Riemann sphere traces out a circle. The farther P moves out along the line, the closer P' gets to the North Pole. Notice, however, that P' never actually arrives at the North Pole.

Now, if we attach ∞ to L to form the generalised circle $L \cup \{\infty\}$ in $\widehat{\mathbb{C}}$, then we can imagine the point P passing through ∞ in the extended complex plane, and the corresponding point P' passing through the North Pole. In this way we see that $L \cup \{\infty\}$ corresponds to an actual circle on $\mathbb{S}$. In fact, we have the following more general result, which we do not prove.

Theorem 1.5

(a) Stereographic projection maps circles on the Riemann sphere $\mathbb{S}$ onto generalised circles in $\widehat{\mathbb{C}}$, and every generalised circle in $\widehat{\mathbb{C}}$ is the image of some circle on $\mathbb{S}$.

(b) Stereographic projection preserves angles.

Remark

It can be shown that the relationship between a point $(u, v, s) \neq (0,0,1)$ on $\mathbb{S}$ and its image $x + iy = \pi(u, v, s)$ is given by

$$\pi(u, v, s) = \left(\frac{u}{1-s}\right) + \left(\frac{v}{1-s}\right)i$$

and

$$\pi^{-1}(x+iy) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

For example, $\pi(0,0,-1) = 0$ and $\pi^{-1}(10i) = (0, 20/101, 99/101)$.

Stereographic projection and cartography

Stereographic projection is one of many projection methods used in cartography to draw the Earth on a flat piece of paper. As a tool for plotting maps, it has the disadvantage that it significantly distorts areas of shapes near the North Pole, which is why it is not used for the most familiar maps of the world. However, angle-preserving projections like stereographic projection are used to create maps for navigation, where direction is more important than area.

Further exercises

Exercise 1.8

(a) Find a linear function that maps the half-plane $\{z : \operatorname{Im} z > 1\}$ onto the half-plane $\{z : \operatorname{Re} z > 2\}$.

(b) Find a linear function that maps the disc $\{z : |z - i| < 1\}$ onto the disc $\{z : |z| < 3\}$.

Exercise 1.9

(a) For each of the following lines and circles, determine whether the image under the reciprocal function is a line or a circle, and determine whether or not it passes through the origin.

(i) $2x - y = 0$ (ii) $x^2 + y^2 = 2$ (iii) $x^2 + (y - 2)^2 = 4$

(iv) $2x + y = 1$

(b) Find an equation for the image for parts (a)(i) and (a)(ii).

Exercise 1.10

Determine how to extend the following functions to functions from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$.

(a) $f(z) = 2z + 6$ (b) $f(z) = \dfrac{1}{z - 1}$

2 Möbius transformations

After working through this section, you should be able to:

- recognise a *Möbius transformation*
- invert a Möbius transformation
- compose two Möbius transformations
- determine the fixed points of a Möbius transformation
- find the Möbius transformation that maps one given set of three points in $\widehat{\mathbb{C}}$ onto another.

2.1 Properties of Möbius transformations

In the previous section you saw that linear functions and the reciprocal function map generalised circles onto generalised circles. In this section we enlarge this collection of circle-preserving functions to a class of functions called *Möbius transformations* (where Möbius is pronounced 'muh-bee-uss' or 'moh-bee-uss').

Definition

A function of the form

$$f(z) = \frac{az+b}{cz+d}, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0,$$

is called a **Möbius transformation**.

Remarks

1. Every linear function is a Möbius transformation, as we can see by setting $c = 0$ and $d = 1$. Also, the reciprocal function $f(z) = 1/z$ is a Möbius transformation with $a = d = 0$ and $b = c = 1$.
2. If $c = 0$, then the transformation f reduces to the function $f(z) = (a/d)z + (b/d)$, with domain $\mathbb{C}$. This is a linear function, because the condition $ad - bc \neq 0$ ensures that both a and d are non-zero, so a/d is also non-zero.

   If $c \neq 0$, then f has domain $\mathbb{C} - \{-d/c\}$, and it has a simple pole at $-d/c$.
3. The condition $ad - bc \neq 0$ is needed to avoid functions like

   $$f(z) = \frac{6z+4}{3z+2},$$

   where the numerator is a constant multiple of the denominator. Such functions are constant throughout their domains, and therefore are not conformal.
4. Notice that we can multiply the numerator and denominator of a Möbius transformation by the same non-zero constant without altering the transformation. For example,

   $$f(z) = \frac{2z+1}{3z+2} \quad \text{and} \quad g(z) = \frac{6z+3}{9z+6}$$

   are the same Möbius transformation.
5. Some texts give Möbius transformations other names, such as *bilinear transformations*, *fractional linear transformations*, *linear fractional transformations* or *homographies*.

The Combination Rules for differentiation show that every Möbius transformation $f(z) = (az+b)/(cz+d)$ is an analytic function. Furthermore, the condition $ad - bc \neq 0$ ensures that the derivative

$$f'(z) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

is non-zero throughout the domain of f. This, in conjunction with Theorem 4.2 of Unit A4, leads to the following theorem.

Theorem 2.1

Every Möbius transformation is analytic and conformal.

As noted earlier, constant functions are not conformal, so they are not Möbius transformations.

You will find the theorem helpful in some parts of the following exercise.

Exercise 2.1

Determine which of the following functions are Möbius transformations.

(a) $f(z) = \dfrac{3}{z}$ (b) $f(z) = \dfrac{3i+2z}{z}$ (c) $f(z) = 1$

(d) $f(z) = z + \dfrac{3}{z}$ (e) $f(z) = \dfrac{1-i+z}{2+3z}$

Next we discuss how to extend the Möbius transformation

$$f(z) = \frac{az+b}{cz+d}, \quad \text{where } ad-bc \neq 0,$$

to $\widehat{\mathbb{C}}$, in the manner described in Subsection 1.3. If $c = 0$, then f is a linear function, so we define

$$f(\infty) = \infty.$$

If $c \neq 0$, then f has a pole at $-d/c$, so we define

$$f(-d/c) = \infty.$$

To find the image of ∞ under f, observe that

$$f(z) = \frac{az+b}{cz+d} = \frac{a+b/z}{c+d/z}.$$

Therefore

$$f(z) \to \frac{a+0}{c+0} = \frac{a}{c} \text{ as } z \to \infty,$$

so we define

$$f(\infty) = a/c.$$

There is no need to remember these formulas for extending f, since it is straightforward to deal with each transformation by inspection as it arises.

Exercise 2.2

Use the conventions above to extend the following Möbius transformations to functions from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$.

(a) $f(z) = \dfrac{2z+i}{-3z+1}$ (b) $f(z) = \dfrac{z-i}{3z+2}$ (c) $f(z) = 2z+1$

From now on we adopt the following convention.

Convention

Each Möbius transformation is considered to be extended to give a function from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$.

This convention holds even when a Möbius transformation is specified by a formula that does not mention the point ∞. For example, we consider the Möbius transformation f specified by the formula

$$f(z) = \frac{z-i}{3z+2}$$

to be the function from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ with rule

$$f(z) = \begin{cases} \dfrac{z-i}{3z+2}, & z \in \mathbb{C} - \{-2/3\}, \\ \infty, & z = -2/3, \\ 1/3, & z = \infty. \end{cases}$$

Let us now investigate the properties of Möbius transformations. We first show that every Möbius transformation $f(z) = (az+b)/(cz+d)$, where $ad - bc \neq 0$, is either a linear function, or a composition of two linear functions and the reciprocal function.

In fact we have already seen that f is a linear function if $c = 0$. If $c \neq 0$, then f can be expressed as a composition of linear functions and the reciprocal function as follows. We can write

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \\ &= \frac{acz+bc}{c(cz+d)} \\ &= \frac{a(cz+d) - ad + bc}{c(cz+d)} \\ &= -\left(\frac{ad-bc}{c}\right)\left(\frac{1}{cz+d}\right) + \frac{a}{c}. \end{aligned}$$

So, for all z in $\mathbb{C} - \{-d/c\}$, we see that $f(z)$ is equal to $(g \circ h \circ k)(z)$, where

$$k(z) = cz + d \quad \text{(linear function)},$$
$$h(z) = \frac{1}{z} \quad \text{(reciprocal function)},$$
$$g(z) = -\left(\frac{ad - bc}{c}\right)z + \frac{a}{c} \quad \text{(linear function)}.$$

Furthermore, we have

$$(g \circ h \circ k)(-d/c) = (g \circ h)(0) = g(\infty) = \infty = f(-d/c)$$

and

$$(g \circ h \circ k)(\infty) = (g \circ h)(\infty) = g(0) = a/c = f(\infty),$$

so

$$f(z) = (g \circ h \circ k)(z), \quad \text{for all } z \in \widehat{\mathbb{C}}.$$

Since Theorem 1.4 guarantees that linear functions and the reciprocal function are one-to-one functions from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$ that map generalised circles onto generalised circles, we obtain the following result.

Theorem 2.2

(a) Möbius transformations are one-to-one mappings from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$.

(b) Möbius transformations map generalised circles onto generalised circles.

It follows from Theorem 2.2(a) that every Möbius transformation has an inverse function. The next example illustrates how to find an inverse function. Observe that the method takes care to avoid performing arithmetic operations with ∞.

Example 2.1

Find the inverse function of the Möbius transformation

$$f(z) = \frac{2z + i}{-3z + 1}.$$

Solution

By Theorem 2.2(a), f is a one-to-one function from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$. Also, $f(1/3) = \infty$ and $f(\infty) = -2/3$. So f maps $\mathbb{C} - \{1/3\}$ onto $\mathbb{C} - \{-2/3\}$.

Now, for each w in $\mathbb{C} - \{-2/3\}$, we need to solve the equation

$$w = \frac{2z + i}{-3z + 1}$$

to obtain a solution z in $\mathbb{C} - \{1/3\}$. We have

$$\begin{aligned} -3zw + w = 2z + i &\iff 3zw + 2z = w - i \\ &\iff z = \frac{w - i}{3w + 2}. \end{aligned}$$

Therefore the inverse function satisfies

$$f^{-1}(w) = \frac{w-i}{3w+2}, \quad \text{for } w \in \mathbb{C} - \{-2/3\}.$$

Since $f(\infty) = -2/3$ and $f(1/3) = \infty$, we have

$$f^{-1}(-2/3) = \infty \quad \text{and} \quad f^{-1}(\infty) = 1/3.$$

Hence the inverse function is

$$f^{-1}(w) = \begin{cases} \dfrac{w-i}{3w+2}, & w \in \mathbb{C} - \{-2/3\}, \\ \infty, & w = -2/3, \\ 1/3, & w = \infty. \end{cases}$$

Exercise 2.3

Find the inverse functions of the following Möbius transformations.

(a) $f(z) = \dfrac{z-i}{3z+2}$ (b) $f(z) = \dfrac{z+2i}{3z-4}$

It appears from Example 2.1 and Exercise 2.3 that the inverse function of a Möbius transformation is itself a Möbius transformation. The next theorem confirms that this is always the case.

Theorem 2.3 Inverse function of a Möbius transformation

The Möbius transformation

$$f(z) = \frac{az+b}{cz+d}, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0,$$

is a one-to-one function from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$ with inverse function

$$f^{-1}(w) = \frac{dw-b}{-cw+a}.$$

The inverse function f^{-1} is itself a Möbius transformation.

Both f and f^{-1} are specified in this theorem for points z and w not equal to ∞. Following our convention, we should interpret these functions as functions from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$ by extending them in the usual way.

To help remember this formula for f^{-1}, you can compare it with the formula for the inverse of the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which is

$$M^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem 2.3 can be proved by the same algebraic argument as was used in Example 2.1 and Exercise 2.3.

Exercise 2.4

Prove Theorem 2.3.

When finding the inverse function of a Möbius transformation f, you may choose to solve the equation $w = f(z)$ for z directly, as in Exercise 2.3. Alternatively, you may prefer to use Theorem 2.3.

Exercise 2.5

Use Theorem 2.3 to check your answers to Exercise 2.3.

Since each Möbius transformation is either a linear function or the composition of two linear functions and the reciprocal function, it is natural to ask whether any new transformations are obtained when Möbius transformations are composed. The following theorem shows that the composition of two Möbius transformations is another Möbius transformation.

Theorem 2.4 Composition of Möbius transformations

Let f and g be Möbius transformations. Then $f \circ g$ is also a Möbius transformation.

Before proving this theorem, we illustrate how to find the composition of two Möbius transformations.

Example 2.2

Find the composition $f \circ g$ of the Möbius transformations

$$f(z) = \frac{iz+1}{2z-2} \quad \text{and} \quad g(z) = \frac{z+i}{2z-1}.$$

Solution

Let z be a complex number that is not a singularity of g and such that $g(z)$ is not a singularity of f. Since $(f \circ g)(z) = f(g(z))$, we have

$$\begin{aligned}
(f \circ g)(z) &= f\left(\frac{z+i}{2z-1}\right) \\
&= \left(i\left(\frac{z+i}{2z-1}\right)+1\right) \Big/ \left(2\left(\frac{z+i}{2z-1}\right)-2\right) \\
&= \frac{i(z+i)+(2z-1)}{2(z+i)-2(2z-1)} = \frac{(2+i)z-2}{-2z+(2+2i)}.
\end{aligned}$$

Hence $f \circ g$ is the Möbius transformation

$$h(z) = \frac{(2+i)z - 2}{-2z + (2+2i)}.$$

Remark

By insisting that z is not a singularity of g and $g(z)$ is not a singularity of f in the solution to Example 2.2, we ensure that none of the algebraic manipulations there involves ∞. We did not show that $(f \circ g)(z) = h(z)$ when z or $g(z)$ is one of these singularities or ∞; however, Theorem 2.4 tells us that $f \circ g$ is a Möbius transformation, so we do have $(f \circ g)(z) = h(z)$ for all points z in $\widehat{\mathbb{C}}$.

Exercise 2.6

Find the composition $f \circ g$ of the Möbius transformations

$$f(z) = \frac{z+2}{z-2} \quad \text{and} \quad g(z) = \frac{z+i}{z-i}.$$

Proof of Theorem 2.4 Let f and g be the Möbius transformations

$$f(z) = \frac{az+b}{cz+d} \quad \text{and} \quad g(z) = \frac{a'z+b'}{c'z+d'},$$

where $ad - bc \neq 0$ and $a'd' - b'c' \neq 0$, and let

$$A = \{z \in \mathbb{C} : z \text{ is a singularity of } g \text{ or } g(z) \text{ is a singularity of } f\}.$$

Arguing in a similar way as we did in Example 2.2, we can show that if $z \in \mathbb{C} - A$, then

$$(f \circ g)(z) = f(g(z)) = h(z),$$

where

$$h(z) = \frac{(aa' + bc')z + (ab' + bd')}{(ca' + dc')z + (cb' + dd')}.$$

To verify that h is a Möbius transformation, we must check the '$ad - bc \neq 0$' condition, which is rather complicated in this case. We have

$$\begin{aligned}
&(aa' + bc')(cb' + dd') - (ab' + bd')(ca' + dc') \\
&\quad = (aca'b' + ada'd' + bcb'c' + bdc'd') - (aca'b' + adb'c' + bca'd' + bdc'd') \\
&\quad = ada'd' + bcb'c' - adb'c' - bca'd' \\
&\quad = (ad - bc)(a'd' - b'c') \neq 0,
\end{aligned}$$

so h is indeed a Möbius transformation.

It remains to prove that $(f \circ g)(z) = h(z)$, for $z \in A \cup \{\infty\}$. Now, Möbius transformations have been defined in such a way that if (z_n) is a sequence in $\mathbb{C}$ that tends to a point z in $\widehat{\mathbb{C}}$, and if k is a Möbius transformation, then

$$k(z_n) \to k(z). \tag{2.1}$$

If z belongs to the finite set $A \cup \{\infty\}$, then we can choose a sequence (z_n) in $\mathbb{C} - A$ such that $z_n \to z$, in which case

$$g(z_n) \to g(z) \quad \text{and} \quad h(z_n) \to h(z).$$

Also, since $g(z_n) \to g(z)$ we can apply equation (2.1) with k replaced by f, z_n replaced by $g(z_n)$ and z replaced by $g(z)$ to give

$$f(g(z_n)) \to f(g(z)).$$

Since $h(z_n) = f(g(z_n))$, we see that $h(z) = f(g(z))$. Hence $(f \circ g)(z) = f(g(z)) = h(z)$, as required. ■

It is useful to observe that the matrix of coefficients of h can be found by multiplying the matrices corresponding to f and g. For instance, for Example 2.2, we have

$$\begin{pmatrix} i & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & i \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2+i & -2 \\ -2 & 2+2i \end{pmatrix}.$$

Another useful property to note is that when we compose any Möbius transformation f with its inverse function f^{-1}, then we obtain a function $f \circ f^{-1}$ that maps every element of $\widehat{\mathbb{C}}$ to itself. This function is the **identity function** on $\widehat{\mathbb{C}}$. It is easy to see that this function is a Möbius transformation, since it can be written as

$$z \longmapsto z = \frac{z+0}{0z+1}.$$

We now have all the properties that are needed to manipulate compositions of Möbius transformations. A summary of these properties is provided by the following theorem. If you have studied group theory, then you will recognise the theorem as asserting that the Möbius transformations form a group under composition of functions.

Theorem 2.5 Group properties

The set of Möbius transformations has the following properties.

Closure If f and g are Möbius transformations, then so is $f \circ g$.

Identity The identity function on $\widehat{\mathbb{C}}$ is a Möbius transformation.

Inverses Each Möbius transformation f has an inverse function f^{-1} which is also a Möbius transformation.

Associativity If f, g and h are Möbius transformations, then

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

The only property that we have not already mentioned is associativity. This is a general property of the composition of functions, which says that it does not matter how we bracket the calculation of a composition of functions. It allows us to write $f \circ g \circ h$ without ambiguity.

August Ferdinand Möbius

Möbius transformations are named after the German astronomer and mathematician August Ferdinand Möbius (1790–1868), who is remembered in mathematics for significant contributions to geometry and number theory. One of his most famous discoveries is the **Möbius strip**, which is a fascinating surface that can be modelled by gluing the two ends of a long, thin strip of paper together after twisting one of the ends through 180° (see Figure 2.1).

Figure 2.1 Möbius strip

His work on Möbius transformations had a great impact on mathematics, after it was taken up by others such as the French mathematician Henri Poincaré (1854–1912), and was shown to be of fundamental importance to deep and far-reaching ideas in geometry.

August Ferdinand Möbius

2.2 The fixed points of a Möbius transformation

In the previous subsection you saw that the identity function on $\widehat{\mathbb{C}}$ is a Möbius transformation. This function sends each point in $\widehat{\mathbb{C}}$ to itself. By contrast we will now show that if f is any Möbius transformation other than the identity function, then it can leave at most two points fixed.

Definition

A **fixed point** of a Möbius transformation f is a point α in the extended complex plane for which $f(\alpha) = \alpha$.

The number of fixed points of the Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

depends on the coefficients a, b, c and d.

If $c = 0$, then f is a linear function, so ∞ is a fixed point. Any other fixed point z of f must satisfy the equation $(az+b)/d = z$, or, equivalently,

$$(d-a)z = b.$$

Therefore if $a \neq d$, then f has exactly one other fixed point $z = b/(d-a)$, and if $a = d$, then f has no other fixed points (unless $b = 0$, in which case f is the identity function).

Suppose now that $c \neq 0$. Then $f(\infty) = a/c \neq \infty$, so all the fixed points of f must lie in $\mathbb{C}$. We can find these fixed points by solving the equation

$$\frac{az+b}{cz+d} = z,$$

that is,

$$cz^2 + (d-a)z - b = 0.$$

Since this is a quadratic equation, it has either one or two solutions in $\mathbb{C}$.

This reasoning leads us to the following theorem.

Theorem 2.6

Each Möbius transformation, other than the identity function, has either one or two fixed points in $\widehat{\mathbb{C}}$.

Exercise 2.7

Determine the fixed points of each of the following Möbius transformations.

(a) $f(z) = \dfrac{z}{2} + 1$ (b) $f(z) = \dfrac{1}{z}$ (c) $f(z) = \dfrac{3z+i}{-iz+3}$

Theorem 2.6 has a number of important implications, including the following theorem.

Theorem 2.7

If two Möbius transformations f and g satisfy $f(z) = g(z)$ for three or more points z in $\widehat{\mathbb{C}}$, then $f = g$.

Proof Suppose that $f(z) = g(z)$, for $z \in A$, where A is a set in $\widehat{\mathbb{C}}$ consisting of three or more points. If $\alpha \in A$, then $f(\alpha) = g(\alpha)$, so

$$(f^{-1} \circ g)(\alpha) = f^{-1}(g(\alpha)) = f^{-1}(f(\alpha)) = \alpha.$$

It follows that the Möbius transformation $f^{-1} \circ g$ fixes each point of A. Since A has at least three points, $f^{-1} \circ g$ is the identity function, by Theorem 2.6. Therefore

$$f = f \circ (f^{-1} \circ g) = (f \circ f^{-1}) \circ g = g,$$

as required. ■

This theorem shows that a Möbius transformation is completely determined by the effect that it has on any three distinct points α, β and γ in $\widehat{\mathbb{C}}$.

Now suppose that we are given two sets $\{\alpha, \beta, \gamma\}$ and $\{\alpha', \beta', \gamma'\}$, each consisting of three distinct points of $\widehat{\mathbb{C}}$. Then it is natural to ask whether we can find a Möbius transformation f that maps

$$\alpha \text{ to } \alpha', \quad \beta \text{ to } \beta' \quad \text{and} \quad \gamma \text{ to } \gamma'.$$

It turns out that we can, but rather than finding such a transformation directly, it is easier to begin by considering the case where α', β', γ' is the **standard triple** of points $0, 1, \infty$. In this case we define f by

$$f(z) = \frac{(z-\alpha)}{(z-\gamma)}\frac{(\beta-\gamma)}{(\beta-\alpha)}. \tag{2.2}$$

Notice that the numerator is not a non-zero multiple of the denominator, because α, β and γ are distinct. Hence f is a Möbius transformation. And clearly $f(\alpha) = 0$, $f(\beta) = 1$ and $f(\gamma) = \infty$, as required.

We have assumed implicitly here that none of the points α, β or γ is ∞ in order for the expression on the right-hand side of formula (2.2) to be valid. However, there is a way in which we can make sense of the formula even when one of the three points is ∞, by taking limits. For example, to find the formula for $f(z)$ when $\beta = \infty$, we observe that

$$\frac{(z-\alpha)}{(z-\gamma)}\frac{(\beta-\gamma)}{(\beta-\alpha)} = \frac{(z-\alpha)}{(z-\gamma)}\frac{(1-\gamma/\beta)}{(1-\alpha/\beta)} \to \frac{z-\alpha}{z-\gamma} \text{ as } \beta \to \infty.$$

Hence

$$f(z) = \frac{z-\alpha}{z-\gamma},$$

and this function satisfies our requirements, because $f(\alpha) = 0$, $f(\infty) = 1$ and $f(\gamma) = \infty$.

A convenient shorthand for this process is to write

$$f(z) = \frac{(z-\alpha)}{(z-\gamma)}\frac{(\infty-\gamma)}{(\infty-\alpha)} = \frac{z-\alpha}{z-\gamma},$$

where we ‘cancel out’ the terms containing ∞. Strictly speaking, the middle step of this equation does not make sense, because it involves arithmetic with ∞. However, it is an illuminating way to explain how to obtain the formula on the right-hand side of the equation, so we allow expressions of this type involving ∞ in these particular circumstances.

A similar approach can be taken if $\alpha = \infty$ or $\gamma = \infty$.

Example 2.3

For each of the triples below, use formula (2.2) to find the Möbius transformation that sends the three given points to the standard triple $0, 1, \infty$.

(a) $i, 1, -i$ (b) $2, \infty, 1+i$

Solution

(a) By formula (2.2), the transformation is

$$f(z) = \frac{(z-i)}{(z+i)}\frac{(1+i)}{(1-i)} = \frac{z-i}{z+i} \times i = \frac{iz+1}{z+i}.$$

(b) Here the transformation is

$$f(z) = \frac{(z-2)}{(z-(1+i))}\frac{(\infty-(1+i))}{(\infty-2)} = \frac{z-2}{z-(1+i)}.$$

Exercise 2.8

For each of the triples below, find the Möbius transformation that sends the three given points to the standard triple $0, 1, \infty$.

(a) $2, 2i, -2$ (b) $i, \infty, 1$ (c) $\infty, 3i, 1$ (d) $1+i, 0, \infty$

With a little practice you can write down the Möbius transformation without reference to formula (2.2). You just need to remember that for α to be mapped to 0, we need $z-\alpha$ in the numerator, and for γ to be mapped to ∞, we need $z-\gamma$ in the denominator. Then you simply multiply by the quantity that yields 1 when β is substituted for z. For instance, in Example 2.3(a), the stages of this process are

$$z-i, \quad \frac{z-i}{z+i}, \quad \frac{1+i}{1-i} \times \frac{z-i}{z+i}.$$

Let us now return to the question that we posed earlier about whether, given two triples of distinct points α, β, γ and α', β', γ', there is a Möbius transformation that maps α to α', β to β' and γ to γ'.

We can construct such a transformation by mapping to and from the standard triple of points $0, 1, \infty$. We know that there is a Möbius transformation f that sends the triple α, β, γ to the standard triple $0, 1, \infty$. Similarly, there is another Möbius transformation g that sends the triple α', β', γ' to the standard triple $0, 1, \infty$. It follows that the composite $g^{-1} \circ f$ must be the required Möbius transformation, since it maps α to α', β to β' and γ to γ' (see Figure 2.2).

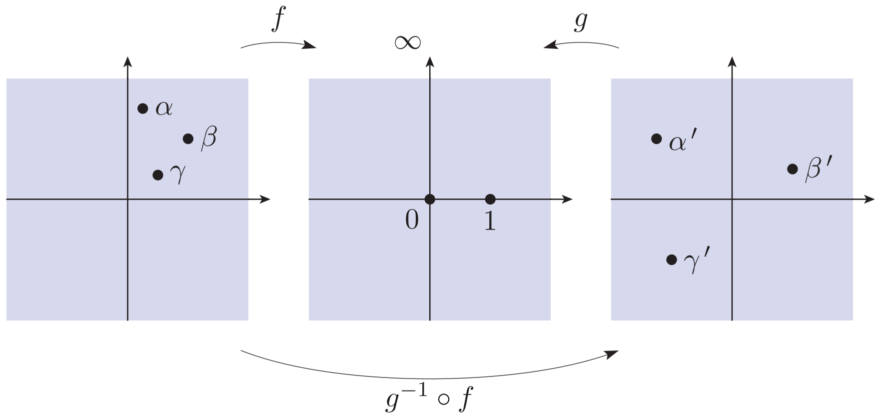


Figure 2.2 Mapping two triples of points to $0, 1, \infty$

In fact, by Theorem 2.7, we know that $g^{-1} \circ f$ is the *only* such Möbius transformation, so we have established the following theorem.

Theorem 2.8

Given two triples of three distinct points α, β, γ and α', β', γ' in $\widehat{\mathbb{C}}$, there is a unique Möbius transformation that maps

$$\alpha \text{ to } \alpha', \quad \beta \text{ to } \beta' \quad \text{and} \quad \gamma \text{ to } \gamma'.$$

The argument used to establish this theorem suggests that if we need to find the Möbius transformation that sends the triple of points α, β, γ to the triple of points α', β', γ', then we must write down the functions f and g, find the inverse of g, and finally form the composite $g^{-1} \circ f$. In fact there is a slightly simpler approach that enables us to carry out the whole calculation in one go. To do this, label the left-hand plane in Figure 2.2 the z-plane and the right-hand plane the w-plane. Then we need to find $w = g^{-1}(f(z))$; we can do so by solving the equation $f(z) = g(w)$ to give w in terms of z. By formula (2.2), we need to solve

$$\frac{(z-\alpha)}{(z-\gamma)}\frac{(\beta-\gamma)}{(\beta-\alpha)} = \frac{(w-\alpha')}{(w-\gamma')}\frac{(\beta'-\gamma')}{(\beta'-\alpha')}.$$

We refer to this equation as the **Implicit Formula for Möbius Transformations** because by solving it to find w in terms of z we obtain a formula for the Möbius transformation that sends α to α', β to β' and γ to γ'. However, when $\alpha' = 0$, $\beta' = 1$ and $\gamma' = \infty$, the equation reduces to

$$w = \frac{(z-\alpha)}{(z-\gamma)}\frac{(\beta-\gamma)}{(\beta-\alpha)},$$

which is the **Explicit Formula for Möbius Transformations** that we found earlier, in formula (2.2).

Theorem 2.9 Implicit and Explicit Formulas for Möbius Transformations

Given two triples of three distinct points α, β, γ and α', β', γ' in $\widehat{\mathbb{C}}$, the unique Möbius transformation that sends α to α', β to β' and γ to γ' is the function f that maps z to w, where

$$\frac{(z-\alpha)}{(z-\gamma)}\frac{(\beta-\gamma)}{(\beta-\alpha)} = \frac{(w-\alpha')}{(w-\gamma')}\frac{(\beta'-\gamma')}{(\beta'-\alpha')}.$$

If $\alpha' = 0$, $\beta' = 1$ and $\gamma' = \infty$, then

$$f(z) = \frac{(z-\alpha)}{(z-\gamma)}\frac{(\beta-\gamma)}{(\beta-\alpha)}.$$

The next example demonstrates how to use the Implicit Formula for Möbius Transformations.

Example 2.4

Find the Möbius transformation that sends the triple of points $-i, -1, i$ to the triple of points $4, 3, 2$.

Solution

We find the transformation by using the Implicit Formula for Möbius Transformations, that is, by solving the equation

$$\frac{(z+i)}{(z-i)}\frac{(-1-i)}{(-1+i)} = \frac{(w-4)}{(w-2)}\frac{(3-2)}{(3-4)}.$$

By evaluating the constant terms and then cross-multiplying, we obtain

$$(z+i)(w-2)i = (w-4)(z-i)(-1),$$

that is,

$$izw - w - 2iz + 2 = -wz + 4z + iw - 4i.$$

On collecting the w terms on the left, we obtain

$$(i+1)zw - (1+i)w = (4+2i)z - (2+4i),$$

so

$$w = \frac{(4+2i)z-(2+4i)}{(1+i)z-(1+i)}.$$

The required transformation is therefore

$$f(z) = \frac{(4+2i)z-(2+4i)}{(1+i)z-(1+i)}.$$

If we wish, we can write this in a simpler form by multiplying the numerator and denominator by $1-i$ to give

$$f(z)=\frac{(6-2i)z-(6+2i)}{2z-2}=\frac{(3-i)z-(3+i)}{z-1}.$$

You may like to check that $f(-i)=4$, $f(-1)=3$ and $f(i)=2$.

Exercise 2.9

Find the Möbius transformation that sends the triple of points $2, 2i, -2$ to the triple of points $i, \infty, 1$.

Compared with the work needed to write down a Möbius transformation that sends three distinct points to the standard triple of points $0, 1, \infty$, the more general calculation, illustrated by Example 2.4 and Exercise 2.9, usually involves a lot more arithmetic. Fortunately, as you will see in Section 4, it is often possible to tackle conformal mapping problems in a way that uses the standard triple of points.

We now give the following important consequence of Theorem 2.8.

Theorem 2.10

Given any two generalised circles C_1 and C_2, there is a Möbius transformation that maps C_1 onto C_2.

Proof Let α, β, γ be any three distinct points on C_1, and let α', β', γ' be any three distinct points on C_2. By Theorem 2.8, there is a Möbius transformation f that maps α, β, γ to α', β', γ', respectively. Since C_1 passes through α, β, γ, it follows that $f(C_1)$ passes through α', β', γ'. But f preserves generalised circles, since it is a Möbius transformation, so $f(C_1)$ is a generalised circle that passes through α', β', γ'. There is only one generalised circle that passes through three distinct points (we do not prove this), so $f(C_1)=C_2$. ■

Theorem 2.8 tells us that there is a *unique* Möbius transformation that maps α to α', β to β' and γ to γ'. In contrast, there are many Möbius transformations that map one generalised circle onto another, as you can see by varying the triple of points α, β, γ on C_1 in the proof of Theorem 2.10: different choices give rise to different Möbius transformations that map C_1 onto C_2.

Further exercises

Exercise 2.10

(a) Determine which of the functions

$$f(z) = \frac{z-i}{iz+1}, \quad g(z) = \frac{z-1}{z+2i} \quad \text{and} \quad h(z) = \frac{z}{2z-i}$$

are Möbius transformations.

(b) Find the inverse function of each of the Möbius transformations that you identified in part (a).

Exercise 2.11

Find the composition $f \circ g$ of the Möbius transformations

$$f(z) = \frac{z-2}{z+i} \quad \text{and} \quad g(z) = \frac{z+1}{z-1}.$$

Exercise 2.12

For each of the following triples, find the Möbius transformation that sends the given three points to the standard triple $0, 1, \infty$.

(a) $1, -1, \infty$ (b) $1, 0, -1$ (c) $1+i, 2-i, 0$

Exercise 2.13

Find the Möbius transformation that sends the triple of points $2i, 1+2i, 1$ to the triple of points $1, 1+i, i$.

Hermann Minkowski

Minkowski spacetime

Minkowski spacetime is a geometric model for unifying space and time in a single four-dimensional system, with three dimensions for space and one for time. It was first conceived by the German mathematician Hermann Minkowski (1864–1909), who was one of the teachers of the physicist Albert Einstein (1879–1955), and it came to play a key role in Einstein's theory of special relatively.

Fundamental to special relativity is the **Lorentz group**, the group of transformations that preserve the properties of Minkowski spacetime, which is named after the Dutch physicist Hendrik Antoon Lorentz (1853–1928). What is remarkable is that the Lorentz group

turns out to be equivalent in a certain sense to the group of Möbius transformations that you have studied in this unit! This correspondence demonstrates that the rich theory of Möbius transformations is reflected in models of deep physical properties of the universe.

3 Images of generalised circles

After working through this section, you should be able to:

- find the image of a generalised circle by using three of its points
- use *substitution* and *inverse points* to find the image of a generalised circle in *Apollonian form*
- find the centre and radius of a circle in Apollonian form
- find inverse points for a given generalised circle.

In the previous section we showed that Möbius transformations preserve generalised circles. However, we have not yet found the image of a generalised circle under a Möbius transformation that is not a linear function or the reciprocal function.

For the reciprocal function, we were able to use Cartesian coordinates to find the image of a generalised circle, but if this approach is used for a general Möbius transformation, then the resulting algebra can be complicated. Instead we will develop three different methods for finding the image of a generalised circle under a Möbius transformation, none of which require Cartesian coordinates.

3.1 The three-point trick

This first method depends on the fact that every generalised circle is completely determined by the positions of any three of its points. If one of these points is ∞, then the generalised circle is the extended line that passes through the other two points. If none of the three points is ∞, and the points are collinear, then again the generalised circle is an extended line. The remaining possibility is that none of the three points is ∞, and the points are not collinear, in which case the generalised circle is the unique circle passing through the three points.

We can determine the image of a generalised circle C under a Möbius transformation by finding the images of three distinct points on C, as the following example demonstrates.

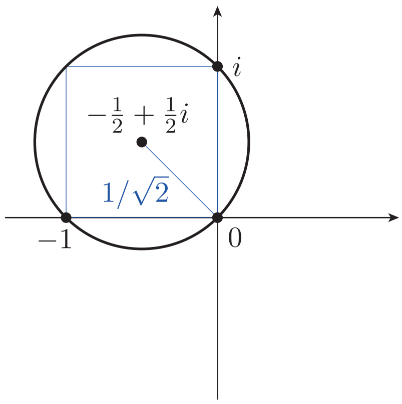


Figure 3.1 Points i, -1 and 0 on the circle centred at $-\frac{1}{2}+\frac{1}{2}i$ of radius $1/\sqrt{2}$

Example 3.1

Find the image of the circle $C = \{z : |z - i| = 1\}$ under the Möbius transformation

$$f(z) = \frac{z - 2i}{z - 2}.$$

Solution

We first pick three distinct points on C. There is no rule about which points to choose, so, to keep the calculations simple, we pick the points 0, $1+i$ and $2i$. Now,

$$f(0) = i, \quad f(1+i) = \frac{1-i}{-1+i} = -1 \quad \text{and} \quad f(2i) = 0.$$

So the image of C is the generalised circle that passes through the points i, -1 and 0. These points are three of the four vertices of a square centred at $-\frac{1}{2}+\frac{1}{2}i$ of side length 1, so they lie on the circle centred at $-\frac{1}{2}+\frac{1}{2}i$ of radius $1/\sqrt{2}$, which passes through these four vertices (see Figure 3.1). Thus $f(C)$ is the circle

$$\left\{z : \left|z - \left(-\tfrac{1}{2}+\tfrac{1}{2}i\right)\right| = 1/\sqrt{2}\right\},$$

as illustrated in Figure 3.2.

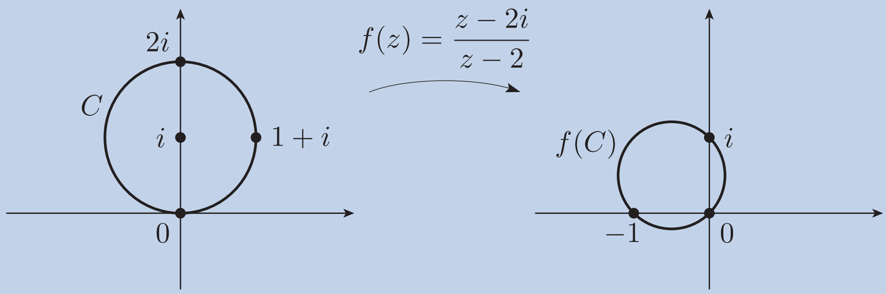


Figure 3.2 Image of the circle $C = \{z : |z - i| = 1\}$ under the Möbius transformation $f(z) = (z - 2i)/(z - 2)$

Remark

Note that $f(i) = -i/(i-2) = (-1+2i)/5$, so the image of the centre of C is *not* the centre of the image circle $f(C)$.

Exercise 3.1

Find the image of the circle $C = \left\{z : |z - (1+i)| = \sqrt{2}\right\}$ under the Möbius transformation

$$f(z) = \frac{-z+2}{z+2}.$$

(*Hint*: Choose the points 0, 2 and $2i$ on C.)

Exactly the same method can be used to find the image of an extended line. Indeed, an extended line is just a special type of generalised circle, so the image can again be found by locating the images of three of its points. Any three points can be used, but the inclusion of the point at infinity often simplifies the calculation, as we see in the following example.

Example 3.2

Find the image of the extended real line $L = \{z : \operatorname{Im} z = 0\} \cup \{\infty\}$ under the Möbius transformation

$$f(z) = \frac{z - i}{z + i}.$$

Solution

To keep the calculations simple, we pick the points 0, 1 and ∞ on L (see Figure 3.3). Then

$$f(0) = -1, \quad f(1) = \frac{1 - i}{1 + i} = -i \quad \text{and} \quad f(\infty) = 1.$$

So the image of L is the generalised circle that passes through the points -1, $-i$ and 1.

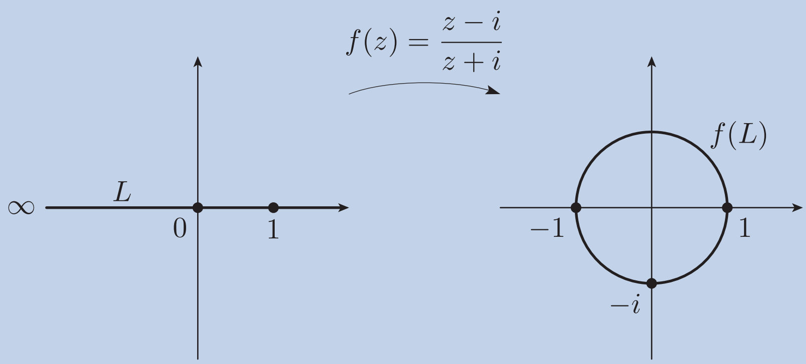


Figure 3.3 Image of the extended real line L under the Möbius transformation $f(z) = (z - i)/(z + i)$

Thus $f(L)$ is the unit circle $\{z : |z| = 1\}$.

Exercise 3.2

Find the image of the extended line $L = \{z : \operatorname{Re} z = \operatorname{Im} z\} \cup \{\infty\}$ under the Möbius transformation

$$f(z) = \frac{z + i}{z - i}.$$

In general it is not straightforward to find the equation of a generalised circle from three points on the circle, so the three-point trick is useful only when the image of the generalised circle is evident from the images of three carefully chosen points, as in Example 3.2.

In the next exercise the pole of the Möbius transformation f lies on the circle, so the image is an extended line, which can be determined from two other image points.

Exercise 3.3

Find the image of the unit circle $C = \{z : |z| = 1\}$ under the Möbius transformation

$$f(z) = \frac{z+1}{z-1}.$$

3.2 The substitution method

In this subsection we will see another method for finding the image of a generalised circle under a Möbius transformation, by *substitution*. This method finds the image of a generalised circle C under a Möbius transformation f by substituting $z = f^{-1}(w)$ into the equation for C. We demonstrate the procedure with an example.

Example 3.3

Find an equation for the image of the unit circle $C = \{z : |z| = 1\}$ under the Möbius transformation

$$f(z) = \frac{iz+i}{-z+1}$$

by substituting $z = f^{-1}(w)$ into the equation for C.

Solution

First observe that $f(C)$ is an extended line, because $1 \in C$ and $f(1) = \infty$. Next, by Theorem 2.3,

$$f^{-1}(w) = \frac{w-i}{w+i}.$$

Now, w is a point on the image $f(C)$ if and only if $f^{-1}(w)$ lies on the unit circle C. That is, $w \in f(C)$ if and only if $w = \infty$ or

$$\left|\frac{w-i}{w+i}\right| = 1.$$

Therefore $f(C)$ consists of ∞ and the set of points satisfied by the equation

$$|w-i| = |w+i|.$$

This equation represents the set of points w that are equidistant from i and $-i$, so it is an equation for the real line (see Figure 3.4). Hence $f(C)$ is the extended real line.

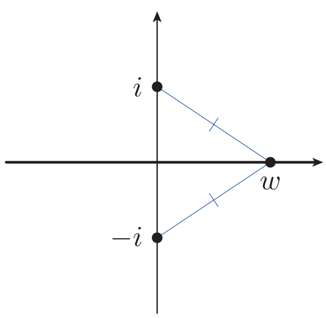


Figure 3.4 w is equidistant from i and $-i$

In general, given any two points α and β in $\mathbb{C}$, the equation

$$|z-\alpha| = |z-\beta|$$

describes the line L that forms the perpendicular bisector of the line segment joining α and β (see Figure 3.5).

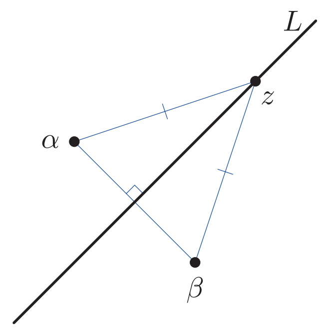


Figure 3.5 z is equidistant from α and β

Conversely, any line L can be described by an equation of the form $|z-\alpha| = |z-\beta|$, where α and β are points that are mirror images of each other in the line. The equation is not unique, since any pair of mirror images can be used to set up the equation.

Exercise 3.4

Find an equation for the image of the unit circle $C = \{z : |z| = 1\}$ under the Möbius transformation

$$f(z) = \frac{z-i}{z+1}$$

by substituting $z = f^{-1}(w)$ into the equation for C.

In the next example, which again uses substitution, we see what happens when the image is a circle.

Example 3.4

Find an equation for the image of the unit circle $C = \{z : |z| = 1\}$ under the Möbius transformation

$$f(z) = \frac{6z-2i}{2z-1}$$

by substituting $z = f^{-1}(w)$ into the equation for C.

Solution

The pole $1/2$ of f does not lie on the unit circle C, so $f(C)$ must be a circle. By Theorem 2.3,

$$f^{-1}(w) = \frac{-w+2i}{-2w+6}.$$

Now, w is a point on the image $f(C)$ if and only if $f^{-1}(w)$ lies on the unit circle C. That is, $w \in f(C)$ if and only if

$$\left|\frac{-w+2i}{-2w+6}\right| = 1.$$

An equation for the image $f(C)$ is therefore

$$|-w+2i| = |-2w+6|,$$

which can be rewritten as

$$|w-2i| = 2|w-3|.$$

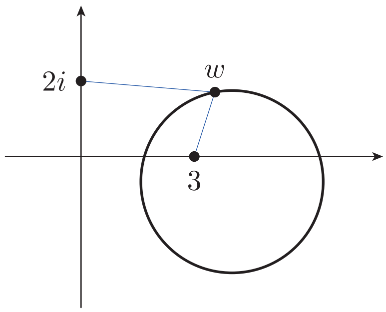


Figure 3.6 w is twice as far from $2i$ as it is from 3

The equation obtained in Example 3.4 is similar to the equation obtained in Example 3.3. The significant difference is the factor 2 that appears in front of the modulus sign on the right. The presence of this factor implies that instead of describing a line, the equation describes the set of points that lie twice as far from $2i$ as they do from 3 (see Figure 3.6). Because Möbius transformations map generalised circles to generalised circles, and this set of points is not an extended line, it must be a circle!

In general, when substitution is used to find the image of an arbitrary generalised circle under a Möbius transformation, it turns out that the resulting equation for the image has the form

$$|z - \alpha| = k|z - \beta|, \quad \text{where } k > 0.$$

Generalised circles that are presented in this form are said to be expressed in **Apollonian form**. The form describes the set of points that lie k times as far from α as they do from β. If $k = 1$, then the equation represents a line, and we will adopt the convention that ∞ is included in the set (so it represents an extended line). If $k \neq 1$, then the equation represents a circle.

In the next subsection you will see how to find the centre and radius of a circle whose equation is expressed in Apollonian form.

Circles of Apollonius

The Ancient Greek geometer Apollonius of Perga (c.262–c.190 BCE) demonstrated geometrically that the set of points that lie k times as far away from α as they do from β is a circle, for $k \neq 1$. Figure 3.7 shows in black a collection of eight circles of the form

$$|z - \alpha| = k|z - \beta|,$$

for various values of k. In the background you can see several feint blue circles, each of which passes through both points α and β. It can be shown that each feint circle intersects each black circle at right angles (they are orthogonal). The collection of all circles of both types is called the **circles of Apollonius**.

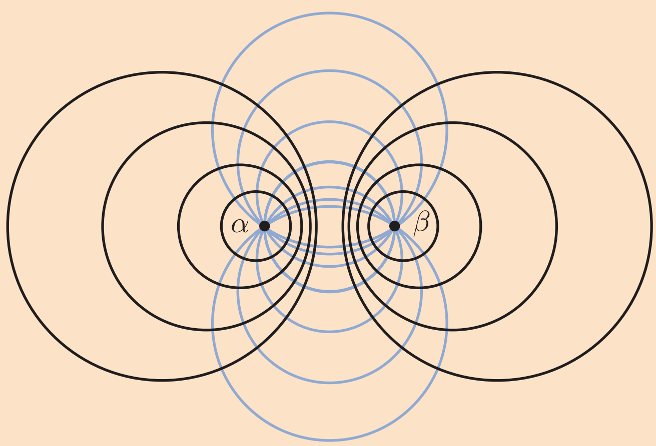


Figure 3.7 Circles of Apollonius

3.3 The inverse points method

So far we have seen two methods to find the image of a generalised circle under a Möbius transformation: the three-point trick and the substitution method. The latter provides an answer in Apollonian form, as does the third method that we introduce here, using *inverse points*. A pair of inverse points is a set of two points that are, in a sense, 'mirror images' of each other with respect to a generalised circle.

To make this more precise, let us first consider two points α and β that are mirror images of each other with respect to an extended line L (see Figure 3.8). Observe that either α and β are equal and lie on L, or L has an equation in Apollonian form $|z-\alpha| = |z-\beta|$. Now, if

$$f(z) = \frac{z-\alpha}{z-\beta},$$

then the equation $|z-\alpha| = |z-\beta|$ can be written in the form $|f(z)| = 1$. But this shows that f maps L into the unit circle (because if $z \in L$ then $|f(z)| = 1$), and in fact f must map L *onto* the unit circle, because f maps generalised circles onto generalised circles. Furthermore, the mirror image points α and β are mapped by f to 0 and ∞, respectively. This suggests the following generalisation of 'mirror images'.

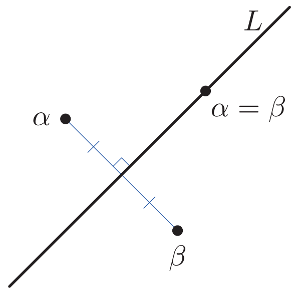


Figure 3.8 Points α and β that are mirror images of each other with respect to L

Definition

Let C be a generalised circle. Then α and β are **inverse points with respect to C** if

- *either* α and β are equal and lie on C
- *or* there exists a Möbius transformation f that maps α to 0, β to ∞, and C onto the unit circle.

The definition is rather unsatisfactory because it does not give us any geometric picture of the relationship between the inverse points α and β and the generalised circle C. We will obtain such a picture shortly (see Figure 3.9, which illustrates a typical pair of inverse points), but first we establish the close connection between inverse points and Apollonian form.

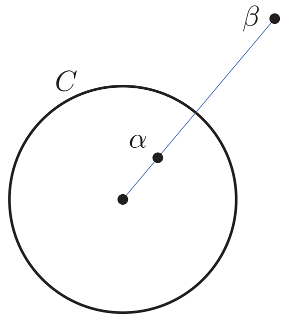


Figure 3.9 Inverse points α and β with respect to a circle C

Theorem 3.1

The points α and β in $\widehat{\mathbb{C}}$ are distinct inverse points with respect to a generalised circle C if and only if

- *either* both α and β belong to $\mathbb{C}$, and C has the equation (in Apollonian form)

$$|z-\alpha| = k|z-\beta|, \quad \text{for some } k > 0$$

- *or* one of the points (β say) is ∞, and C has the equation

$$|z-\alpha| = r, \quad \text{for some } r > 0.$$

You may choose to omit the proof of Theorem 3.1 on a first reading.

Proof First suppose that α and β are distinct inverse points with respect to a generalised circle C. Then, by definition, there is a Möbius transformation f that maps α to 0, β to ∞, and C onto the unit circle.

Let γ be the point on C that f maps to 1. Since α and β are distinct, they cannot lie on C. It follows that γ cannot be equal to α or β. Since f is completely determined by the effect that it has on α, β and γ, we have

$$f(z) = \frac{(z-\alpha)}{(z-\beta)}\frac{(\gamma-\beta)}{(\gamma-\alpha)},$$

with the usual rules about 'cancelling terms involving ∞' if one of α, β and γ happens to be ∞. But f maps C onto the unit circle, so an equation for C is $|f(z)| = 1$, that is,

$$\left|\frac{(z-\alpha)}{(z-\beta)}\frac{(\gamma-\beta)}{(\gamma-\alpha)}\right| = 1. \tag{3.1}$$

If $\alpha, \beta \in \mathbb{C}$, then this becomes

$$|z-\alpha| = k|z-\beta|, \quad \text{where } k = \left|\frac{\gamma-\alpha}{\gamma-\beta}\right| > 0.$$

However, if one of the points α or β is ∞ (β say), then the $(\gamma-\beta)$ term in equation (3.1) 'cancels' with the $(z-\beta)$ term to give

$$|z-\alpha| = r, \quad \text{where } r = |\gamma-\alpha| > 0.$$

Conversely, suppose that C has equation $|z-\alpha| = k|z-\beta|$, with $\alpha, \beta \in \mathbb{C}$ and $k > 0$. If α and β were equal, then the equation $|z-\alpha| = k|z-\beta|$ would reduce to $k = 1$, which is not an equation for a generalised circle. Hence $\alpha \neq \beta$. Next let

$$f(z) = \frac{1}{k}\left(\frac{z-\alpha}{z-\beta}\right).$$

Then f is a Möbius transformation that maps α to 0, β to ∞, and C onto the unit circle (since $|f(z)| = 1$), so α and β are inverse points with respect to the generalised circle C.

Suppose now that C has equation $|z-\alpha| = r$, and $\beta = \infty$. Let

$$f(z) = \frac{z-\alpha}{r}.$$

Once again, f maps α to 0, β to ∞, and C onto the unit circle, so α and β are inverse points with respect to C. ■

If C is an ordinary circle centred at α, then C has equation $|z-\alpha| = r$, for some $r > 0$. It follows from Theorem 3.1 that α and ∞ are inverse points with respect to C. Thus it is easy to identify at least one pair of inverse points of an ordinary circle.

Corollary

The centre α of a circle C and the point ∞ are inverse points with respect to C.

The next theorem says, informally speaking, that Möbius transformations preserve inverse points.

Theorem 3.2

Let f be a Möbius transformation. If α and β are inverse points with respect to a generalised circle C, then $f(\alpha)$ and $f(\beta)$ are inverse points with respect to $f(C)$.

Proof If α and β are distinct inverse points with respect to the generalised circle C, then, by the definition of inverse points, there must be a Möbius transformation, g say, that maps α to 0, β to ∞, and C onto the unit circle. Thus $g \circ f^{-1}$ maps $f(\alpha)$ to 0, $f(\beta)$ to ∞, and $f(C)$ onto the unit circle. It follows that $f(\alpha)$ and $f(\beta)$ are inverse points with respect to $f(C)$.

On the other hand, if α and β are equal and lie on C, then $f(\alpha)$ equals $f(\beta)$ and this point lies on $f(C)$, so once again $f(\alpha)$ and $f(\beta)$ are inverse points with respect to $f(C)$. ■

Remark

An interesting consequence of Theorem 3.2 is the following observation. Suppose that C is an ordinary circle, and f is a Möbius transformation such that $f(C)$ is also an ordinary circle. We know from the corollary to Theorem 3.1 that the centre of C and the point ∞ are inverse points with respect to C, and the centre of $f(C)$ and the point ∞ are inverse points with respect to $f(C)$. Since Möbius transformations preserve inverse points, we see that f maps the centre of C to the centre of $f(C)$ if and only if $f(\infty) = \infty$, that is, if and only if f is a linear function.

We can use Theorem 3.2 to provide us with our final method (the inverse points method) for finding the image of a generalised circle C under a Möbius transformation.

Example 3.5

Find an equation for the image of the circle $C = \{z : |z - 2i| = 3\}$ under the Möbius transformation

$$f(z) = \frac{iz + 5}{z + i}.$$

Solution

Since C has centre $2i$, it follows from the corollary to Theorem 3.1 that $2i$ and ∞ are inverse points with respect to C. Now,

$$f(2i) = \frac{3}{3i} = -i \quad \text{and} \quad f(\infty) = i.$$

So, by Theorem 3.2, $-i$ and i must be inverse points with respect to $f(C)$.

Hence, by Theorem 3.1, $f(C)$ has an equation of the form

$$|w + i| = k|w - i|, \quad \text{for some } k > 0.$$

Since $5i$ lies on C, it follows that $f(5i) = 0$ lies on $f(C)$, so

$$k = \frac{|0 + i|}{|0 - i|} = 1.$$

Hence $f(C)$ is the extended real line

$$\{w : |w + i| = |w - i|\} \cup \{\infty\}.$$

In the solution to Example 3.5 we could choose any point on C to determine the value of k; choosing the point $5i$ just makes the calculation simple.

The same method can be used to find an equation for the image of any circle C. The resulting equation for the image of C is normally in Apollonian form, but if one of the inverse points of C maps to ∞, and the other maps to α say, then the equation will have the form $\{z : |z - \alpha| = r\}$. In such cases the value of r can be evaluated in the same way as k, by substituting a point from the image into the equation.

Exercise 3.5

For each of the following circles C, find the image of C under the Möbius transformation

$$f(z) = \frac{z - 1}{z - i}.$$

(a) $C = \{z : |z - (1 + i)| = \sqrt{2}\}$

(b) $C = \{z : |z - 1| = 1\}$

(c) $C = \{z : |z - i| = 1\}$

(d) $C = \{z : |z - (1 + i)| = 1\}$

Because the inverse points method usually yields an equation for the image in Apollonian form, we still may not know where the centre of the image is located (if the image is indeed a circle). The next theorem can be used to find the centre and radius of any circle written in Apollonian form.

Theorem 3.3

Let C be the generalised circle with equation

$$|z-\alpha| = k|z-\beta|, \quad \text{where } \alpha, \beta \in \mathbb{C} \text{ and } k > 0.$$

(a) If $k \neq 1$, then C is the circle centred at λ of radius r, where

$$\lambda = \frac{\alpha - k^2\beta}{1-k^2} \quad \text{and} \quad r = \frac{k|\alpha-\beta|}{|1-k^2|}.$$

Also, λ lies on the line through α and β, and

$$(\alpha-\lambda)\overline{(\beta-\lambda)} = r^2.$$

(b) If $k = 1$, then C is the extended line through $\frac{1}{2}(\alpha+\beta)$ that is perpendicular to the line through α and β.

Theorem 3.3(a) provides us with the geometric interpretation of inverse points that we promised earlier. It tells us that in the case $k \neq 1$, the centre λ of C lies on the line through α and β, because

$$\lambda = t\alpha + (1-t)\beta, \quad \text{where } t = \frac{1}{1-k^2} \in \mathbb{R}. \tag{3.2}$$

Furthermore, since $(\alpha-\lambda)\overline{(\beta-\lambda)} = r^2$, we have

$$|\alpha-\lambda||\beta-\lambda| = |(\alpha-\lambda)\overline{(\beta-\lambda)}| = r^2.$$

So the product of the distances of α and β from the centre of C is equal to the square of the radius of C (see Figure 3.10).

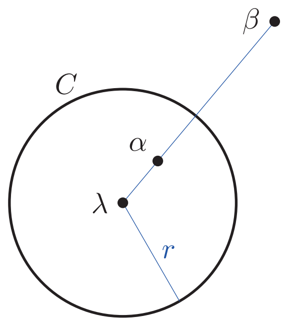


Figure 3.10 The points α, β, λ satisfy $|\alpha-\lambda||\beta-\lambda| = r^2$

In the special case where C is the unit circle $|z| = 1$, we have $\lambda = 0$. Thus if α and β are inverse points with respect to C, then $\alpha\overline{\beta} = 1$, that is, $\alpha = 1/\overline{\beta}$. Hence α is the reciprocal of the mirror image of β in the real axis (see Figure 3.11). The function $f(z) = 1/\overline{z}$, which sends β to α, differs from the reciprocal function $f(z) = 1/z$ in that, like reflections, it is not analytic.

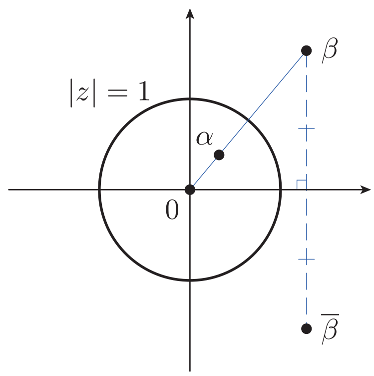


Figure 3.11 The points α, β satisfy $\alpha = 1/\overline{\beta}$

Before proving Theorem 3.3 we ask you to use it in the following exercise.

Exercise 3.6

Determine the centres and radii of the circles determined by the following equations.

(a) $|z-i| = \sqrt{2}\,|z-1|$ (b) $|z-1| = \sqrt{2}\,|z|$

Proof of Theorem 3.3

(a) Suppose that $k \neq 1$. The circle centred at λ of radius r, where

$$\lambda = \frac{\alpha - k^2\beta}{1-k^2} \quad \text{and} \quad r = \frac{k|\alpha - \beta|}{|1-k^2|},$$

has equation $|z-\lambda| = r$, that is,

$$\left| z - \frac{\alpha - k^2\beta}{1-k^2} \right| = \frac{k|\beta - \alpha|}{|1-k^2|}.$$

By writing $\beta - \alpha = (z-\alpha) - (z-\beta)$, and multiplying through by $|1-k^2|$, we obtain

$$|(z-\alpha) - k^2(z-\beta)| = k|(z-\alpha) - (z-\beta)|. \tag{3.3}$$

Squaring the left-hand side of this equation, and expanding (using $|w|^2 = w\overline{w}$), gives

$$|z-\alpha|^2 - k^2(z-\alpha)\overline{(z-\beta)} - k^2\,\overline{(z-\alpha)}(z-\beta) + k^4|z-\beta|^2,$$

and squaring and expanding the right-hand side gives

$$k^2|z-\alpha|^2 - k^2(z-\alpha)\overline{(z-\beta)} - k^2\,\overline{(z-\alpha)}(z-\beta) + k^2|z-\beta|^2.$$

Hence we can rewrite equation (3.3) in the form

$$|z-\alpha|^2 + k^4|z-\beta|^2 = k^2|z-\alpha|^2 + k^2|z-\beta|^2.$$

Rearranging this gives $(1-k^2)|z-\alpha|^2 = k^2(1-k^2)|z-\beta|^2$, which simplifies to

$$|z-\alpha| = k|z-\beta|.$$

In this way we see that the equations $|z-\alpha| = k|z-\beta|$ and $|z-\lambda| = r$ are equivalent, when $k \neq 1$, so C does indeed have centre λ and radius r. Furthermore, λ lies on the line through α and β, by equation (3.2), and we also have

$$\begin{aligned}
(\alpha-\lambda)\overline{(\beta-\lambda)} &= \left(\alpha - \frac{\alpha-k^2\beta}{1-k^2}\right)\overline{\left(\beta - \frac{\alpha-k^2\beta}{1-k^2}\right)} \\
&= \left(\frac{k^2(\beta-\alpha)}{1-k^2}\right)\overline{\left(\frac{\beta-\alpha}{1-k^2}\right)} \\
&= \left(\frac{k|\alpha-\beta|}{|1-k^2|}\right)^2 \\
&= r^2.
\end{aligned}$$

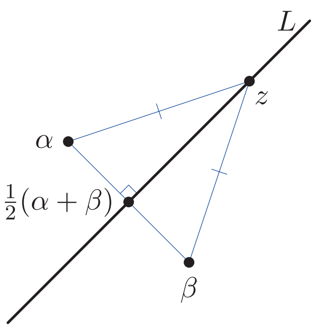


Figure 3.12 z is equidistant from α and β

(b) Suppose that $k = 1$. The equation $|z-\alpha| = |z-\beta|$ represents the extended line L, where L is the perpendicular bisector of the line through α and β, together with the point ∞. This line contains the point $\frac{1}{2}(\alpha+\beta)$, as illustrated in Figure 3.12. ■

Any circle has infinitely many pairs of inverse points, as the following theorem shows.

Theorem 3.4 Existence of inverse points

Let C be a generalised circle, and let β be an arbitrary point of $\widehat{\mathbb{C}}$. Then there is a unique point α such that α and β are inverse points with respect to C.

Proof Suppose first that C is the unit circle $|z| = 1$. If $\beta \in C$, then it has a unique inverse point $\alpha = \beta$ with respect to C. If $\beta = \infty$, then it has a unique inverse point $\alpha = 0$, and if $\beta = 0$, then it has a unique inverse point $\alpha = \infty$. Let us assume, then, that $\beta \notin C$ and $\beta \neq 0, \infty$.

Consider the circle $|z - \alpha| = k|z - \beta|$, where $\alpha = 1/\overline{\beta}$ and $k = 1/|\beta|$. By Theorem 3.3, this circle has centre and radius

$$\lambda = \frac{1/\overline{\beta} - \beta/|\beta|^2}{1 - 1/|\beta|^2} = 0 \quad \text{and} \quad r = \frac{(1/|\beta|)\,|1/\overline{\beta} - \beta|}{|1 - 1/|\beta|^2|} = \frac{|\beta|\,|1/\overline{\beta} - \beta|}{||\beta|^2 - 1|} = 1,$$

so it is the unit circle C. Thus, by Theorem 3.1, α and β are inverse points with respect to C. Furthermore, if γ and β are inverse points with respect to C, then they must satisfy $\gamma\overline{\beta} = 1$, by Theorem 3.3, so $\gamma = 1/\overline{\beta}$ is the *unique* inverse point of β with respect to C.

Now suppose that C is any generalised circle and $\beta \in \widehat{\mathbb{C}}$. Let f be a Möbius transformation that maps the unit circle onto C, and let α' be the unique inverse point of $\beta' = f^{-1}(\beta)$ with respect to the unit circle. Then, by Theorem 3.2, $\alpha = f(\alpha')$ is the unique inverse point of $\beta = f(\beta')$ with respect to C. ■

Since any generalised circle has infinitely many pairs of inverse points, it follows that any generalised circle also has infinitely many Apollonian forms. For a simple example, the extended imaginary axis L can be represented by the equation $|z - 1| = |z + 1|$, because it is the set of points that are equidistant from 1 and -1 (together with ∞). But L can also be represented by the equation $|z - 2| = |z + 2|$, or in fact $|z - t| = |z + t|$, for any non-zero real number t.

Let us now see how to use inverse points to transform equations of generalised circles into Apollonian form.

Example 3.6

Determine the point α such that α and $\beta = 1 + 2i$ are inverse points with respect to the circle $C = \{z : |z - i| = 2\}$.

Hence write down an equation for C in Apollonian form.

Solution

Here C has centre i and radius 2, so $\lambda = i$ and $r = 2$. Using the equation $(\alpha - \lambda)\overline{(\beta - \lambda)} = r^2$ from Theorem 3.3, we see that

$$\alpha - i = \frac{2^2}{\overline{\beta - i}} = \frac{4}{\overline{1 + i}} = \frac{4(1 + i)}{2} = 2 + 2i.$$

Thus $\alpha = 2 + 3i$, so C has an equation of the form

$$|z - (2 + 3i)| = k|z - (1 + 2i)|.$$

Since $-i$ lies on C, we find that

$$k = \frac{|-i - (2 + 3i)|}{|-i - (1 + 2i)|} = \frac{|-2 - 4i|}{|-1 - 3i|} = \frac{\sqrt{20}}{\sqrt{10}} = \sqrt{2}.$$

Hence an equation for C in Apollonian form is

$$|z - (2 + 3i)| = \sqrt{2}\,|z - (1 + 2i)|.$$

Exercise 3.7

Determine the point α such that α and $\beta = 1 + i$ are inverse points with respect to the circle $C = \{z : |z - 2| = 1\}$.

Hence write down an equation for C in Apollonian form.

In the case of an extended line, the inverse points can be found by reflection.

Example 3.7

Determine the point α such that α and i are inverse points with respect to the extended line $L = \{z : \operatorname{Re} z = \operatorname{Im} z\} \cup \{\infty\}$.

Hence write down an equation for L in Apollonian form.

Solution

The reflection of the point i in L is 1, so $\alpha = 1$ (see Figure 3.13). An equation for L is therefore

$$|z - 1| = |z - i|.$$

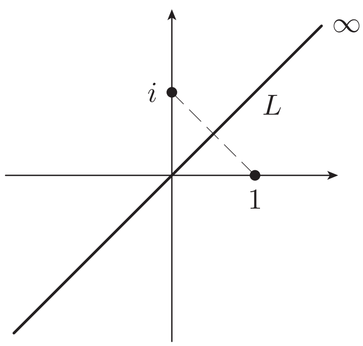


Figure 3.13 i and 1 are inverse points with respect to $L = \{z : \operatorname{Re} z = \operatorname{Im} z\} \cup \{\infty\}$

Exercise 3.8

(a) Determine the point α such that α and $2 + 3i$ are inverse points with respect to the extended imaginary axis L, and hence write down an equation for L in Apollonian form.

(b) Determine the point α such that α and $4 - 2i$ are inverse points with respect to the extended line $L = \{z : \operatorname{Re} z = 3\} \cup \{\infty\}$, and hence write down an equation for L in Apollonian form.

To finish this section we summarise the strategies we have seen for finding images of generalised circles under Möbius transformations.

Strategies for finding images of generalised circles under Möbius transformations

- The three-point trick
- The substitution method
- The inverse points method

If you are given three (or more) points on a generalised circle, but you do not know an equation for the generalised circle, then you should use the three-point trick to find the image.

If you are given an equation for the generalised circle, then you may be able to find the image using the three-point trick with three carefully chosen points. Otherwise, use either the substitution method or the inverse points method to find the image.

Further exercises

Exercise 3.9

Use the three-point trick to find the image of the circle $C = \{z : |z| = 2\}$ under each of the following Möbius transformations.

(a) $f(z) = \dfrac{z+2i}{z-2}$ (b) $f(z) = \dfrac{z-2}{z}$

Exercise 3.10

For each of the following generalised circles C, find the image of C under the Möbius transformation

$$f(z) = \frac{z-i}{z+i}.$$

Give your answers in Apollonian form, where appropriate, and state whether the images are circles or extended lines.

(a) $C = \{z : |z+i| = 1\}$ (b) $C = \{z : |z-i| = 1\}$

Exercise 3.11

Find the centre and radius of each of the following circles.

(a) $\{z : |z-i| = 2|z+3i|\}$ (b) $\{z : |z| = 6|z+i|\}$

Exercise 3.12

(a) Determine the point α such that α and $\beta = 1+i$ are inverse points with respect to each of the following circles.

(i) $C_1 = \{z : |z| = 2\}$ (ii) $C_2 = \left\{z : |z-i| = \frac{1}{2}\right\}$

(b) Write down equations for the circles in part (a) in Apollonian form.

4 Transforming regions

After working through this section, you should be able to:

- write down the *boundary* in $\widehat{\mathbb{C}}$ of a set that lies in $\widehat{\mathbb{C}}$
- find the image of a *generalised open disc* under a Möbius transformation
- construct a Möbius transformation that sends one generalised open disc onto another
- construct a Möbius transformation that sends a *lune* onto another lune of the same angle
- construct a composite mapping that sends one given region onto another
- understand the definition of the inverse sin and tan functions.

4.1 Images of generalised open discs

So far we have concentrated on finding the images of generalised circles under Möbius transformations. However, one of the main objectives of the unit is to investigate the effect that conformal mappings have on regions. We begin this investigation by describing how to find the images of some basic regions under Möbius transformations. The technique is to find the image of the boundary of the region and use this to determine the image of the region itself.

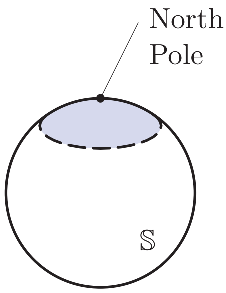


Figure 4.1 An open disc centred at the North Pole

First we need to generalise the definition of 'boundary' that we gave in Subsection 5.1 of Unit A3 to make it applicable to sets in the *extended* complex plane. We do this by introducing the idea of an **open disc centred at ∞**. Such a set has the form $\{z : |z| > M\} \cup \{\infty\}$, where $M > 0$, and on the Riemann sphere $\mathbb{S}$ it corresponds to a cap-shaped disc, centred at the North Pole (see Figure 4.1). This enables us to make the following definitions.

Definitions

Let A be a subset of $\widehat{\mathbb{C}}$, and let $\alpha \in \widehat{\mathbb{C}}$. Then α is a **boundary point** in $\widehat{\mathbb{C}}$ of A if each open disc centred at α contains at least one point of A and at least one point of $\widehat{\mathbb{C}} - A$.

The set of boundary points in $\widehat{\mathbb{C}}$ of A forms the **boundary** in $\widehat{\mathbb{C}}$ of A.

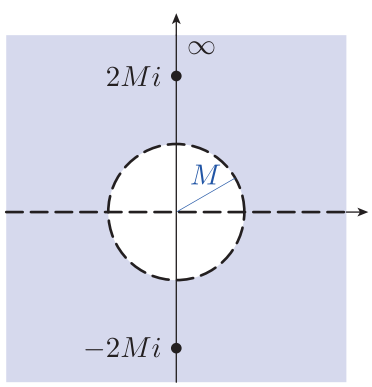


Figure 4.2 The open disc $\{z : |z| > M\} \cup \{\infty\}$

If A is a bounded set, then the boundary in $\widehat{\mathbb{C}}$ of A is the same as the boundary of A in $\mathbb{C}$. For example, the boundary in $\widehat{\mathbb{C}}$ of the open unit disc is the unit circle.

To find the boundary in $\widehat{\mathbb{C}}$ of a set that is not bounded, we must check whether the point at infinity lies on the boundary in $\widehat{\mathbb{C}}$. For example, the boundary in $\widehat{\mathbb{C}}$ of the upper half-plane $\{z : \operatorname{Im} z > 0\}$ includes ∞ because every open disc of the form $\{z : |z| > M\} \cup \{\infty\}$ centred at ∞ (see Figure 4.2) includes both the point $2Mi$, which belongs to the upper half-plane, and the point $-2Mi$, which does not.

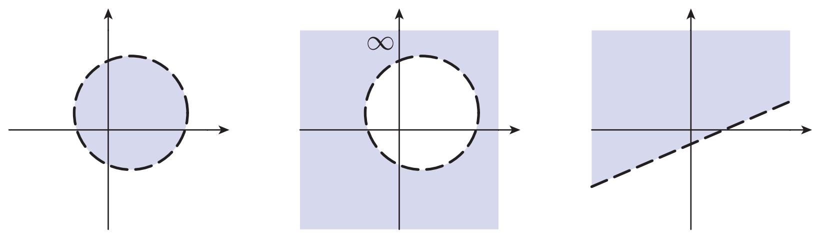
Exercise 4.1

Write down the boundary in $\widehat{\mathbb{C}}$ of each of the following subsets of $\widehat{\mathbb{C}}$.

(a) $\{z : |z| < 2\}$ (b) $\{z : \operatorname{Re} z < 1\}$ (c) $\{z : |z| > 3\}$

(d) $\{z : |z| > 3\} \cup \{\infty\}$

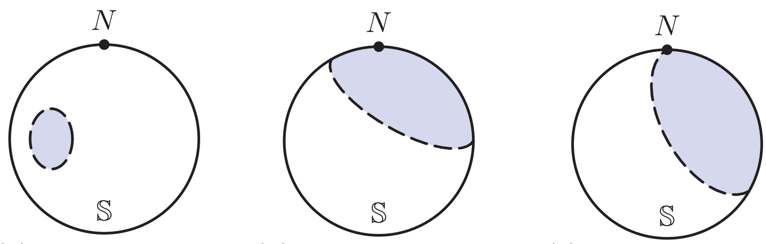
The sets in parts (a), (b) and (d) of Exercise 4.1 have boundaries in $\widehat{\mathbb{C}}$ that are generalised circles. Each generalised circle separates $\widehat{\mathbb{C}}$ into two parts, which together form the complement in $\widehat{\mathbb{C}}$ of the generalised circle. Each of these parts is called a **generalised open disc**. Each generalised open disc is the inside of a circle, the outside of a circle *together with the point* ∞, or an open half-plane, as illustrated in Figure 4.3.



(a) (b) (c)

Figure 4.3 Generalised open discs: (a) the inside of a circle, (b) the outside of a circle together with ∞, (c) an open half-plane

On the Riemann sphere, generalised open discs correspond to cap-shaped discs, as shown in Figure 4.4 (in which (a), (b) and (c) illustrate the same type of generalised open discs as those in Figure 4.3(a), (b) and (c), respectively, and where N denotes the North Pole).



(a) (b) (c)

Figure 4.4 Generalised open discs on the Riemann sphere $\mathbb{S}$

We saw in Theorem 1.4 that Möbius transformations map generalised circles onto generalised circles, so it is perhaps unsurprising that they also map generalised open discs onto generalised open discs.

Theorem 4.1

Let f be a Möbius transformation, and let D be a generalised open disc with boundary C in $\widehat{\mathbb{C}}$. Then $f(D)$ is a generalised open disc with boundary $f(C)$ in $\widehat{\mathbb{C}}$.

Theorem 4.1 is illustrated in Figure 4.5.

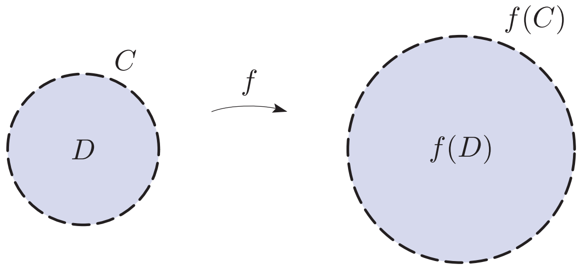


Figure 4.5 Image of a generalised open disc D under a Möbius transformation f

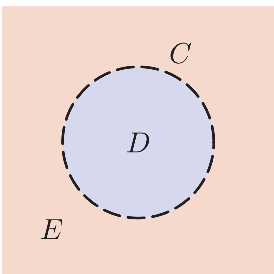


Figure 4.6 Two generalised open discs D and E with common boundary C in $\widehat{\mathbb{C}}$

Proof Let E be the generalised open disc other than D that lies in the complement of C in $\widehat{\mathbb{C}}$ and has boundary C in $\widehat{\mathbb{C}}$ (see Figure 4.6). Then $\widehat{\mathbb{C}} = D \cup C \cup E$, and because f is a one-to-one mapping from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$, we see that $\widehat{\mathbb{C}} = f(D) \cup f(C) \cup f(E)$. Here $f(C)$ is a generalised circle, since C is a generalised circle.

Now, D is a connected set (possibly together with the point ∞), so $f(D)$ is also connected (possibly together with ∞), because it is the continuous image of a connected set (Theorem 4.2 of Unit A3). Hence $f(D)$ must lie in one of the two generalised open discs that form the complement of $f(C)$ in $\widehat{\mathbb{C}}$, and $f(E)$ must (for similar reasons) lie in the other. Since $\widehat{\mathbb{C}} = f(D) \cup f(C) \cup f(E)$, we see that $f(D)$ must in fact be equal to one of these generalised open discs in the complement of $f(C)$. These two generalised open discs both have boundary $f(C)$ in $\widehat{\mathbb{C}}$, as required. ■

The next example demonstrates how Theorem 4.1 can be used to determine the image of a generalised open disc.

Example 4.1

Find the image of the open unit disc $D = \{z : |z| < 1\}$ under the Möbius transformation

$$f(z) = \frac{z - i}{z + 1}.$$

Solution

The boundary of D is the unit circle $C = \{z : |z| = 1\}$. By Exercise 3.4, we know that the image of this circle under f is the extended line $L = \{z : \operatorname{Re} z = -\operatorname{Im} z\} \cup \{\infty\}$. By Theorem 4.1, $f(D)$ must be one of the two generalised open discs

$$\{z : \operatorname{Re} z < -\operatorname{Im} z\} \quad \text{and} \quad \{z : \operatorname{Re} z > -\operatorname{Im} z\}$$

that have boundary L in $\widehat{\mathbb{C}}$. To decide which one, we pick a point in D, such as 0, and observe that $f(0) = -i$. Since $-i$ lies in $\{z : \operatorname{Re} z < -\operatorname{Im} z\}$, it follows that $f(D) = \{z : \operatorname{Re} z < -\operatorname{Im} z\}$, as illustrated in Figure 4.7.

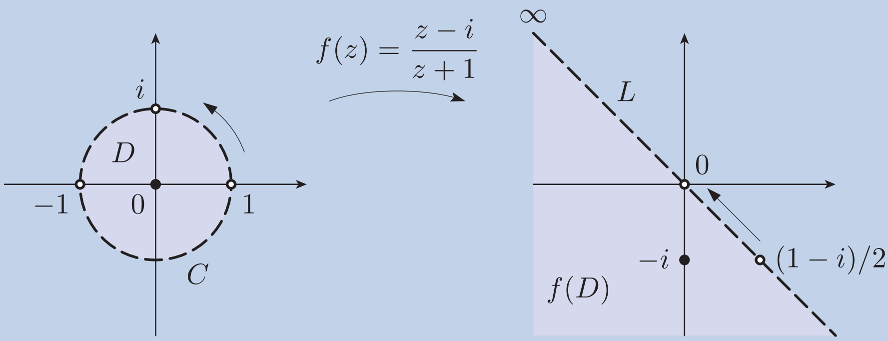


Figure 4.7 Image of the open unit disc D under $f(z) = (z-i)/(z+1)$

Remark

In Example 4.1 we had to decide which of the two regions with boundary L in $\widehat{\mathbb{C}}$ was equal to $f(D)$, and we did this by considering the image of a point of D under f. Another way to make this decision is to observe that if a point z moves anticlockwise around the unit circle C passing through 1, i and -1, in that order, then the image point $f(z)$ passes through $f(1) = (1-i)/2$, $f(i) = 0$ and $f(-1) = \infty$, in that order along L, moving leftwards and upwards. The open unit disc D lies to the left of the point z as it traverses C, and, correspondingly, the region $f(D)$ should lie to the left of $f(z)$ as it traverses L. It follows that $f(D)$ is the half-plane to the left of L, as shown in Figure 4.7 (the directions in which the paths are traversed are marked in that figure by arrows).

In this way, we have used the *orientation* of the boundary paths to determine the image of the region D. We will sometimes make use of arguments involving orientation to guide our reasoning.

The next exercise follows on from Exercise 3.3.

Exercise 4.2

Find the image of the open unit disc $D = \{z : |z| < 1\}$ under the Möbius transformation

$$f(z) = \frac{z+1}{z-1}.$$

The following example demonstrates how to find the image of the outside of a circle under a Möbius transformation.

Example 4.2

Find the image of the generalised open disc

$$D = \{z : |z - 2i| > 3\} \cup \{\infty\}$$

under the Möbius transformation

$$f(z) = \frac{iz+5}{z+i}.$$

Hence find the image of the region $\mathcal{R} = \{z : |z - 2i| > 3\}$ under f.

Solution

The boundary in $\widehat{\mathbb{C}}$ of D is the circle $C = \{z : |z - 2i| = 3\}$. In Example 3.5 we showed that the image of C under f is the extended real line. It follows from Theorem 4.1 that $f(D)$ is one of the two generalised open discs that have boundary in $\widehat{\mathbb{C}}$ the extended real line. Hence $f(D)$ is either the upper or the lower half-plane. Since $\infty \in D$, and $f(\infty) = i$, we see that $f(D)$ is the upper half-plane.

Furthermore, we have that $\mathcal{R} = D - \{\infty\}$, so

$$f(\mathcal{R}) = f(D) - \{f(\infty)\} = \{w : \operatorname{Im} w > 0\} - \{i\},$$

as illustrated in Figure 4.8.

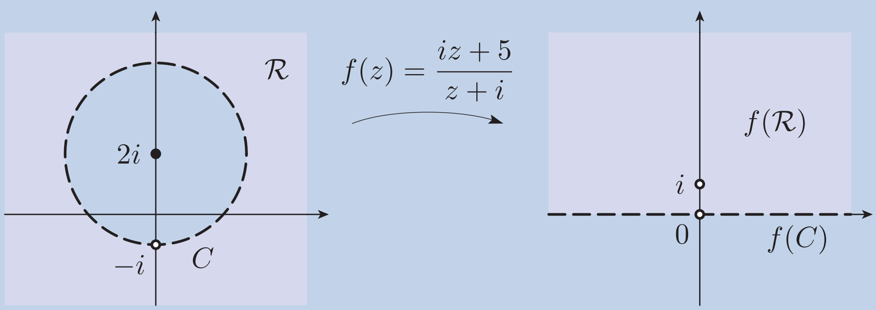


Figure 4.8 Image of $\mathcal{R} = \{z : |z - 2i| > 3\}$ under $f(z) = (iz+5)/(z+i)$

Exercise 4.3

Use Exercise 3.5(c) to find the image of the open disc $D = \{z : |z - i| < 1\}$ under the Möbius transformation

$$f(z) = \frac{z-1}{z-i}.$$

Hence find the image of the punctured open disc $\mathcal{R} = \{z : 0 < |z - i| < 1\}$ under f.

4.2 Conformal mappings between basic regions

The techniques illustrated in Examples 4.1 and 4.2 can be used to find the image of any generalised open disc under a Möbius transformation. Often, however, given any two generalised open discs, we need to find a Möbius transformation that maps one disc onto the other disc. More generally, given any two regions in the plane, we would like to know whether there is a one-to-one conformal mapping from one onto the other. The remainder of this unit is concerned with just this question.

We begin by looking at some conformal mappings from certain 'basic regions' onto the right half-plane, shown in Figure 4.9. These basic regions are of various types: other half-planes, discs, sectors and strips. Then in the next subsection we will compose conformal mappings of basic regions to obtain more complicated mappings.

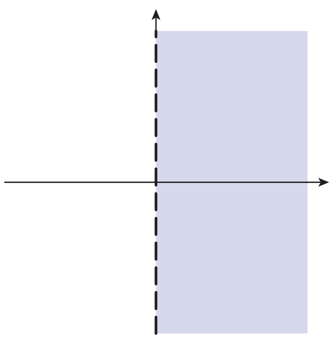


Figure 4.9 Right half-plane

Throughout we will use the notation $\mathcal{R}$ and $\mathcal{S}$ for regions, often asking for a conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

Our first example is about finding a conformal mapping from one open half-plane onto another. Since any two half-planes have the same shape, we can make use of a linear function.

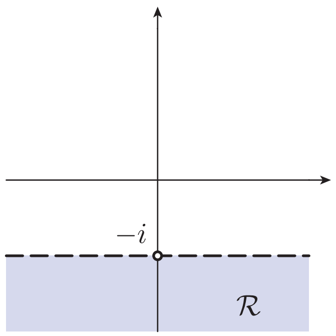


Figure 4.10 The open half-plane $\mathcal{R} = \{z : \operatorname{Im} z < -1\}$

Example 4.3

Find a one-to-one conformal mapping from the open half-plane

$$\mathcal{R} = \{z : \operatorname{Im} z < -1\},$$

illustrated in Figure 4.10, onto the right half-plane $\mathcal{S} = \{z : \operatorname{Re} z > 0\}$.

Solution

The two half-planes have the same shape, so we can apply a linear function to map one onto the other. To do this, first rotate $\mathcal{R}$ anticlockwise through an angle of $\pi/2$ about the origin, and then translate the resulting half-plane by one unit to the left. The rotation is given by the function

$$z \longmapsto e^{i\pi/2}z, \quad \text{that is,} \quad z \longmapsto iz.$$

The translation is given by the function

$$z \longmapsto z - 1.$$

Composing these mappings, we obtain the function

$$f(z) = iz - 1,$$

which is a linear function that maps $\mathcal{R}$ onto S. Since linear functions are one-to-one conformal mappings on the entire complex plane, we see that f is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

The final line of Example 4.3 says that 'f is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$', meaning that the *restriction* of f to $\mathcal{R}$ is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$. This is a convenient shorthand.

It is helpful to think of $\mathcal{R}$ as lying in the z-plane and $\mathcal{S}$ as lying in the w-plane. The function

$$f(z) = iz - 1$$

specifies a point in the w-plane in terms of a point in the z-plane, so we can write the function in the alternative form

$$w = iz - 1.$$

You will see in Subsection 4.3 that this type of notation is particularly convenient when several mappings have to be composed. Another advantage of this notation is that we can often calculate the rule for the inverse function f^{-1} by rearranging the equation to obtain z in terms of w.

In this case we find that

$$z = -i(w + 1).$$

In other words, $f^{-1}(w) = -i(w + 1)$.

The function f and its inverse are illustrated (with the alternative notation) in Figure 4.11.

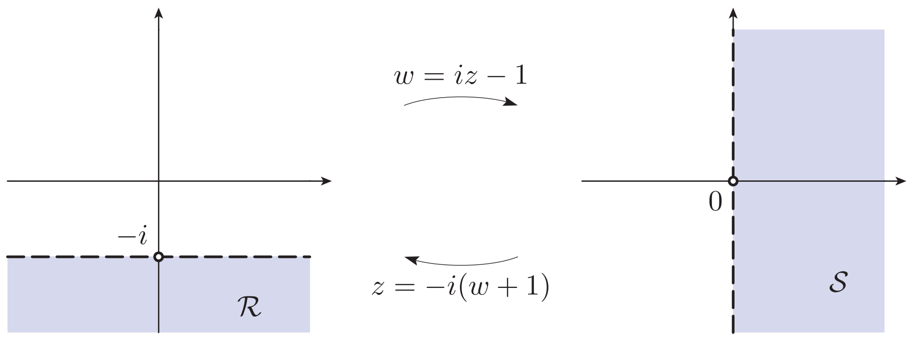


Figure 4.11 Conformal mapping from the open half-plane $\mathcal{R} = \{z : \operatorname{Im} z < -1\}$ onto the open half-plane $\mathcal{S} = \{w : \operatorname{Re} w > 0\}$

Linear functions can also be used to map other regions of the same shape onto each other, such as discs.

Exercise 4.4

Find a one-to-one conformal mapping from the open unit disc

$$\mathcal{R} = \{z : |z| < 1\}$$

onto the open disc

$$\mathcal{S} = \{z : |z - 3i| < 2\}.$$

Remark

It is important to appreciate that the conformal mappings obtained in Example 4.3 and Exercise 4.4 are *not* unique, and nor are any of the conformal mappings between regions that we obtain in this unit. In Exercise 4.4, for example, we could first apply any rotation g about the origin, which is a one-to-one conformal mapping from the open unit disc to itself, before applying the conformal mapping f from $\mathcal{R}$ onto $\mathcal{S}$. The composition $f \circ g$ is also a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$, no matter what angle of rotation for g we choose.

Next we consider conformal mappings from open discs to open half-planes. Both types of region are generalised open discs, so, as we saw in the previous subsection, we can make use of Möbius transformations.

Example 4.4

Find a one-to-one conformal mapping from the open unit disc

$$\mathcal{R} = \{z : |z| < 1\}$$

onto the right half-plane $\mathcal{S} = \{z : \operatorname{Re} z > 0\}$.

Solution

The boundary of $\mathcal{R}$ is the unit circle, and the boundary in $\widehat{\mathbb{C}}$ of $\mathcal{S}$ is the extended imaginary axis $\{z : \operatorname{Re} z = 0\} \cup \{\infty\}$. We will construct a Möbius transformation f that sends three points on the unit circle to three points on the extended imaginary axis. By the three-point trick, f will map the unit circle onto the extended imaginary axis, which is a good start.

We can choose any two triples of points that we like, but we may as well make our choices as simple as possible. So we choose f to satisfy

$$f(1) = \infty, \quad f(i) = i \quad \text{and} \quad f(-1) = 0,$$

as illustrated in Figure 4.12.

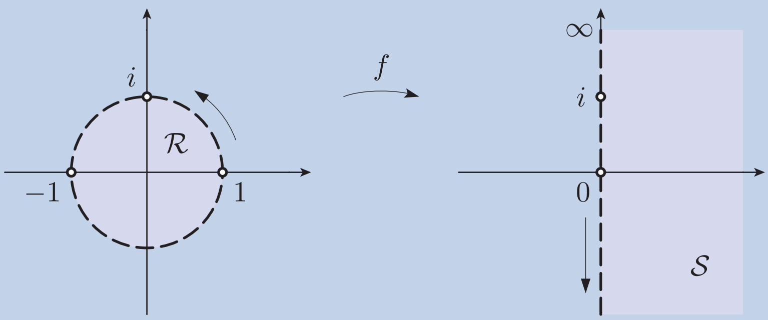


Figure 4.12 Conformal mapping from the open unit disc $\mathcal{R}$ to the right half-plane $\mathcal{S}$

We can find a formula for f using the Implicit Formula for Möbius Transformations, which gives

$$\frac{(z-1)}{(z+1)}\frac{(i+1)}{(i-1)} = \frac{(w-\infty)}{(w-0)}\frac{(i-0)}{(i-\infty)}.$$

Simplifying this, we obtain

$$\frac{z-1}{z+1} \times (-i) = \frac{i}{w}.$$

Hence

$$w = -\frac{z+1}{z-1} = \frac{z+1}{-z+1}.$$

This function $f(z) = (z+1)/(-z+1)$ maps the unit circle onto the extended imaginary axis, and since $0 \in \mathcal{R}$ and $f(0) = 1$, we deduce from Theorem 4.1 that $f(\mathcal{R})$ is the right half-plane $\mathcal{S}$.

Since Möbius transformations are one-to-one conformal mappings on $\widehat{\mathbb{C}}$, we see that f is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

Remark

Notice that as a point z traverses the unit circle anticlockwise, passing through 1, i and -1 in that order, the region $\mathcal{R}$ lies to the left of z. Accordingly, as the image point $f(z)$ traverses the imaginary axis downwards, through ∞, i and 0 in that order, the region $\mathcal{S}$ lies to the left of $f(z)$ (see Figure 4.12). Because we chose the two triples of points in this way, we found that the unit disc $\mathcal{R}$ does indeed map onto the right half-plane $\mathcal{S}$ under f, rather than mapping onto the left half-plane (as in Exercise 4.2).

We can find the inverse function of the conformal mapping obtained in Example 4.4 by rearranging the equation

$$w = \frac{z+1}{-z+1}$$

to find z in terms of w (or use Theorem 2.3). We obtain

$$z = \frac{w-1}{w+1}.$$

The mapping and its inverse are illustrated in Figure 4.13.

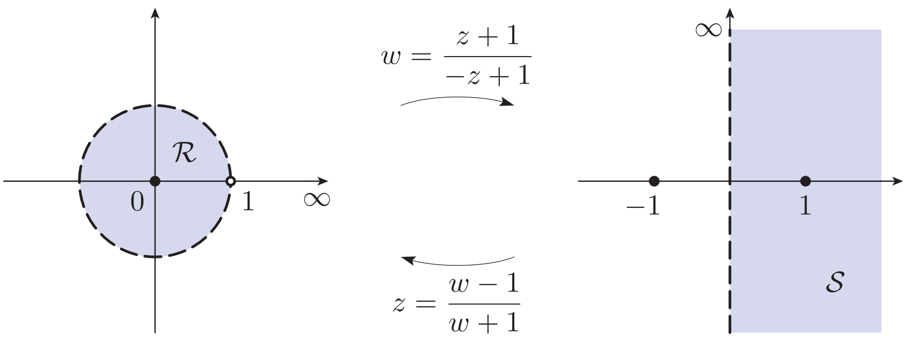


Figure 4.13 Conformal mapping from the open unit disc onto the right half-plane

Exercise 4.5

Find a one-to-one conformal mapping from the open disc

$$\mathcal{R} = \{z : |z-2| < 1\}$$

onto the upper half-plane $\mathcal{S} = \{z : \operatorname{Im} z > 0\}$.

In Example 4.4 we found a Möbius transformation that maps one generalised open disc onto another by choosing two sets of three points on the boundaries of each of the discs. There is an alternative approach to finding a suitable Möbius transformation: instead of choosing two such sets of three points, we choose a pair of inverse points with respect to each boundary, and *one* point on the boundary of each generalised open disc.

For instance, in Example 4.4 we could choose the point 1 on the boundary of the unit disc, and the points 0 and ∞, which are inverse points with respect to the unit circle. We also choose the point ∞ on the boundary of the right half-plane, and the points 1 and -1, which are inverse points with respect to the extended imaginary axis. These points are marked in Figure 4.13. The unique Möbius transformation f that satisfies

$$f(0) = 1, \quad f(\infty) = -1 \quad \text{and} \quad f(1) = \infty$$

is then a one-to-one conformal mapping from the open unit disc onto the right half-plane. We obtain the same transformation as that of Example 4.4.

Let us demonstrate the method in more detail with another example, this time of a mapping from an open half-plane onto the open unit disc. The inverse points strategy outlined above is particularly effective when mapping onto the open unit disc because, as we have observed already, 0 and ∞ are inverse points with respect to the unit circle, and the point 1 lies on the unit circle, so we can use this strategy with the Explicit Formula for Möbius Transformations to find the required Möbius transformation.

Example 4.5

Find a one-to-one conformal mapping from the open half-plane

$$\mathcal{R} = \{z : \operatorname{Im} z < \operatorname{Re} z\}$$

onto the open unit disc $\mathcal{S} = \{z : |z| < 1\}$.

Solution

The points 1 and i are inverse points with respect to the extended line L that is the boundary of $\mathcal{R}$, and $\infty \in L$ (see Figure 4.14). Also, 0 and ∞ are inverse points with respect to the unit circle, and the point 1 lies on the unit circle. So let us choose a Möbius transformation f that satisfies

$$f(1) = 0, \quad f(i) = \infty \quad \text{and} \quad f(\infty) = 1.$$

Using the Explicit Formula for Möbius Transformations, we obtain

$$f(z) = \frac{(z-1)}{(z-i)}\frac{(\infty - i)}{(\infty - 1)} = \frac{z-1}{z-i}.$$

Now, 1 and i are inverse points with respect to L, so 0 and ∞ are inverse points with respect to $f(L)$, by preservation of inverse points under Möbius transformations. Therefore $f(L)$ is a circle centred at 0.

Since $1 \in f(L)$, we see that $f(L)$ is the unit circle. Observe also that $1 \in \mathcal{R}$ and $f(1) = 0$ lies in the open unit disc. Therefore we can apply Theorem 4.1 to see that the Möbius transformation f is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

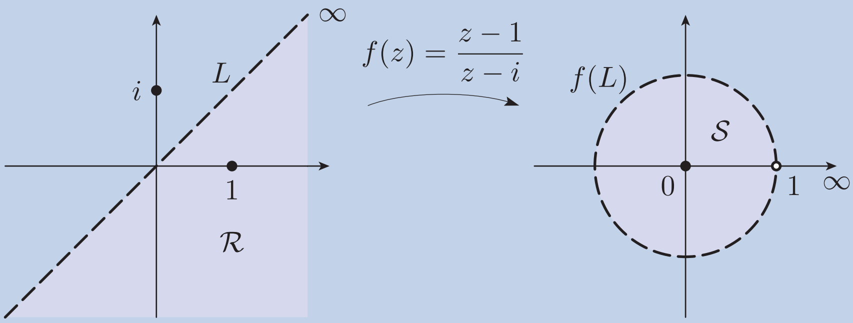


Figure 4.14 Conformal mapping from the open half-plane $\mathcal{R} = \{z : \operatorname{Im} z < \operatorname{Re} z\}$ onto the open unit disc $\mathcal{S}$

Exercise 4.6

Find a one-to-one conformal mapping from the open half-plane

$$\mathcal{R} = \{z : \operatorname{Re} z < 1\}$$

onto the open unit disc $\mathcal{S} = \{z : |z| < 1\}$.

Before we continue with our next example, it is useful to make the following observation.

Observation

Any one-to-one analytic mapping from a region $\mathcal{R}$ onto a region $\mathcal{S}$ is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

This observation is true because if f is a one-to-one analytic mapping from $\mathcal{R}$ onto $\mathcal{S}$, then we can see from the corollary to the Local Mapping Theorem (Theorem 3.2 of Unit C2) that $f'(z) \neq 0$, for all $z \in \mathcal{R}$. Hence f is conformal, by Theorem 4.2 of Unit A4.

We use the observation in the next example to help us construct a conformal mapping from an open sector to an open half-plane.

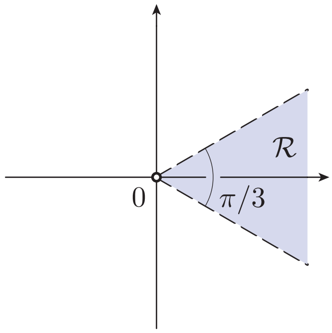


Figure 4.15 The open sector $\mathcal{R} = \{z : |\operatorname{Arg} z| < \pi/6\}$

Example 4.6

Find a one-to-one conformal mapping from the open sector

$$\mathcal{R} = \{z : |\operatorname{Arg} z| < \pi/6\},$$

illustrated in Figure 4.15, onto the right half-plane $\mathcal{S} = \{z : \operatorname{Re} z > 0\}$.

Solution

Informally speaking, the method for mapping $\mathcal{R}$ onto $\mathcal{S}$ is to straighten out the corner at the vertex of the sector by opening up the arms of the sector to an angle of π. Both arms will then lie in a line and the sector will become a half-plane.

The simplest way to do this is to apply a suitable power function. The angle at the vertex of $\mathcal{R}$ is $\pi/3$, so we need to open up the angle by a factor of 3. We do this by applying the cube function

$$f(z) = z^3.$$

This function cubes the modulus of each complex number and triples the argument. Therefore it is a one-to-one function from

$$\{z : |\operatorname{Arg} z| < \pi/6\} \quad \text{onto} \quad \{w : |\operatorname{Arg} w| < \pi/2\}.$$

Since f is analytic, we see from the observation preceding the example that it is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

A little care is needed to obtain the inverse function from the half-plane $\mathcal{S}$ to the sector $\mathcal{R}$ in Example 4.6 because, in contrast to earlier examples, the cube function is not one-to-one on $\mathbb{C}$. However, all the points in the sector $\mathcal{R}$ have arguments between $-\pi/6$ and $\pi/6$, so the sector lies in the image set of the principal cube root function. We can therefore use the principal cube root function to map $\mathcal{S}$ back onto $\mathcal{R}$, as illustrated in Figure 4.16.

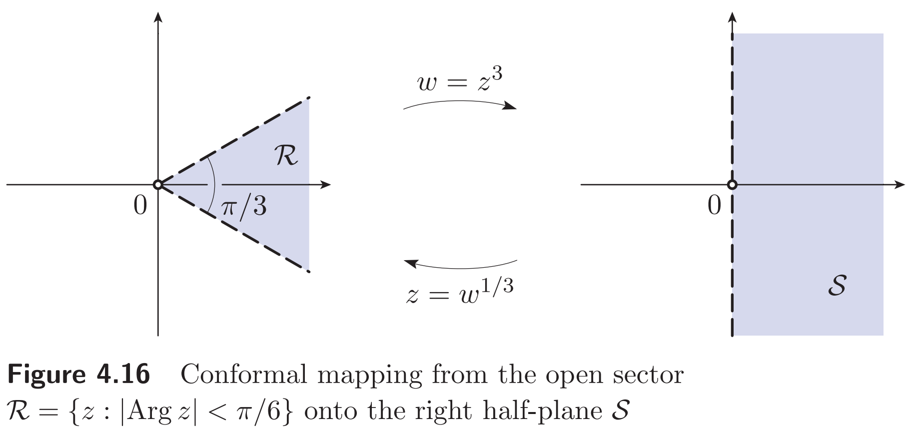


Figure 4.16 Conformal mapping from the open sector $\mathcal{R} = \{z : |\operatorname{Arg} z| < \pi/6\}$ onto the right half-plane $\mathcal{S}$

Sometimes, however, we need to use nth roots other than principal nth roots to find inverse functions of power functions. For example, suppose that we wish to find a one-to-one conformal mapping from the open sector

$$\mathcal{R} = \{z : \pi/2 < \operatorname{Arg} z < 5\pi/6\},$$

illustrated in Figure 4.17, onto the right half-plane $\mathcal{S}$.

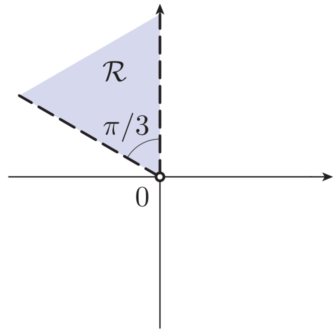


Figure 4.17 The open sector $\mathcal{R} = \{z : \pi/2 < \operatorname{Arg} z < 5\pi/6\}$

Once again the cube function $f(z) = z^3$ is a suitable mapping because it triples arguments, so it maps $\mathcal{R} = \{z : \pi/2 < \operatorname{Arg} z < 5\pi/6\}$ onto the set of points w with argument between $3\pi/2$ and $5\pi/2$. Since $3\pi/2 = -\pi/2 + 2\pi$ and $5\pi/2 = \pi/2 + 2\pi$, we see that f maps $\mathcal{R}$ onto the set

$$\{w : -\pi/2 < \operatorname{Arg} w < \pi/2\},$$

which is the right half-plane $\mathcal{S}$.

This time the principal cube root function $z = w^{1/3}$ is *not* the inverse function of $f(z) = z^3$ $(z \in \mathcal{R})$ because, as we saw in Figure 4.16, this function maps $\mathcal{S}$ onto the open sector

$$\{z : |\operatorname{Arg} z| < \pi/6\}.$$

Notice, however, that we can map this sector onto $\mathcal{R}$ by using the mapping $z \longmapsto e^{2\pi i/3}z$, which rotates points anticlockwise about 0 through the angle $2\pi/3$. It follows that the composite function $z = e^{2\pi i/3}w^{1/3}$ is a conformal mapping from $\mathcal{S}$ onto $\mathcal{R}$. Furthermore, this function satisfies

$$f(e^{2\pi i/3}w^{1/3}) = (e^{2\pi i/3}w^{1/3})^3 = e^{2\pi i}w = w,$$

so it is the inverse function of f, as illustrated in Figure 4.18. It is *a* cube root function, even though it is not the *principal* cube root function.

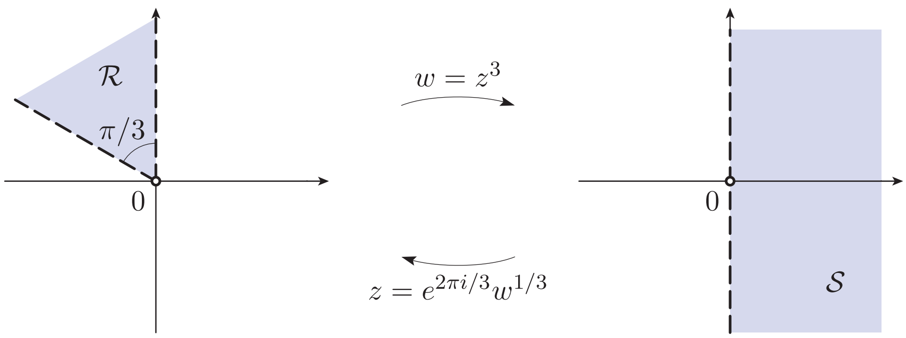


Figure 4.18 Conformal mapping from the open sector $\mathcal{R} = \{z : \pi/2 < \operatorname{Arg} z < 5\pi/6\}$ onto the right half-plane $\mathcal{S}$

Exercise 4.7

Find a one-to-one conformal mapping f from the lower-left quadrant

$$\mathcal{R} = \{z : -\pi < \operatorname{Arg} z < -\pi/2\},$$

illustrated in Figure 4.19, onto the upper half-plane $\mathcal{S} = \{z : \operatorname{Im} z > 0\}$.

Determine the rule for the inverse function f^{-1}.

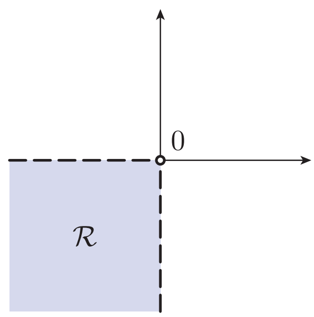


Figure 4.19 Lower-left quadrant

A sector is a region bounded by two intersecting lines. Let us now turn our attention to strips; that is, regions bounded by two *parallel* lines. In Example 5.1 of Unit A2 you saw that, when restricted to the horizontal strip

$$\{x + iy : -\pi < y \leq \pi\},$$

the exponential function is a one-to-one function with image set $\mathbb{C} - \{0\}$ and inverse function the principal logarithm function. Reasoning in a similar way to that example, we can show that exp restricted to the open horizontal strip

$$\mathcal{R} = \{x + iy : -\pi/2 < y < \pi/2\}$$

is a one-to-one function with image set the right half-plane. (To do this, write $w = e^{x+iy} = e^x e^{iy}$, with $-\pi/2 < y < \pi/2$.) This function is a conformal mapping because exp is conformal throughout $\mathbb{C}$.

The mapping and its inverse function are illustrated in Figure 4.20. The image of the line $\{x + iy : y = b\}$, where $-\pi/2 < b < \pi/2$, is the ray $\{se^{ib} : s > 0\}$, in agreement with Figure 4.5 of Unit A2.

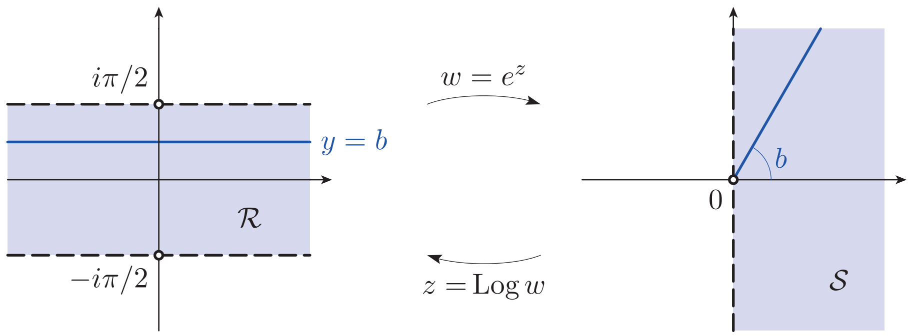
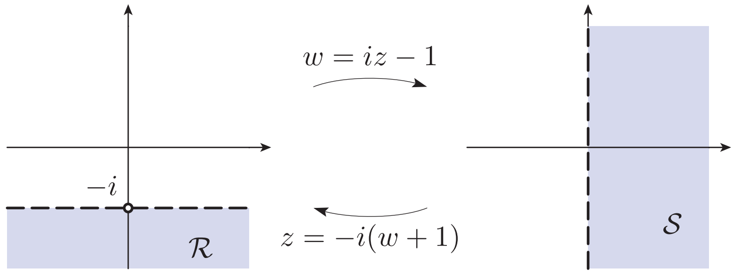
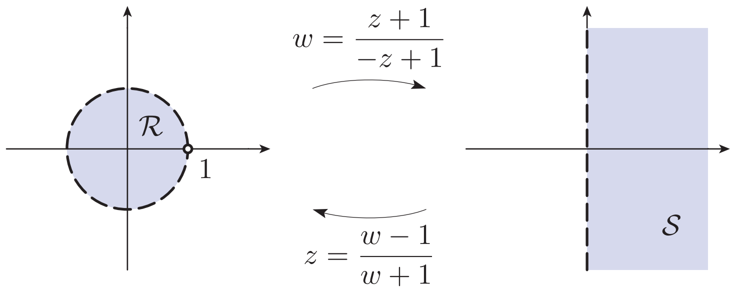
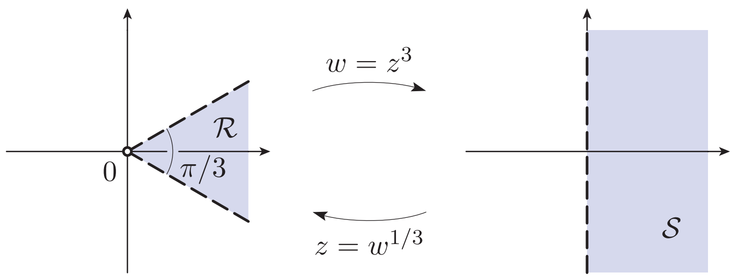
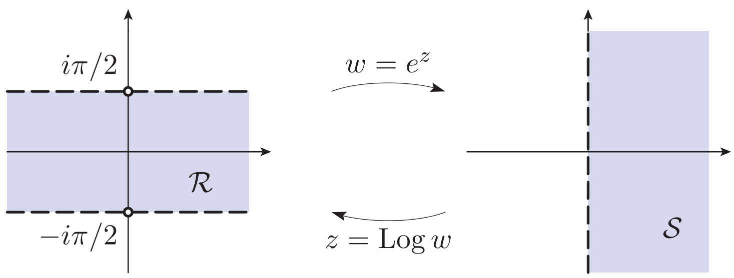


Figure 4.20 Conformal mapping from the horizontal strip $\mathcal{R} = \{x + iy : -\pi/2 < y < \pi/2\}$ onto the right half-plane $\mathcal{S}$

In general, the exponential function maps any horizontal strip of width less than 2π to a sector with vertex at the origin, and the angle at the vertex equals the width of the strip. For the example above, the strip has width π, so the sector has angle π, which implies that it is a half-plane.

We have now seen a number of basic one-to-one conformal mappings from certain basic regions onto the right half-plane. For convenience, we collect these examples together in a table.

Table 4.1 Conformal mappings from basic regions onto the right half-plane

Basic region	Mapping	Example
Open half-plane	Linear function	
Open disc	Möbius transformation	
Open sector with vertex at the origin	Power function	
Open horizontal strip	Exponential function	

Of the four regions $\mathcal{R}$ in Table 4.1, the first two are generalised open discs, and each of the last two (the open sector and the open horizontal strip) is the intersection of two generalised open discs, as demonstrated in Figure 4.21.

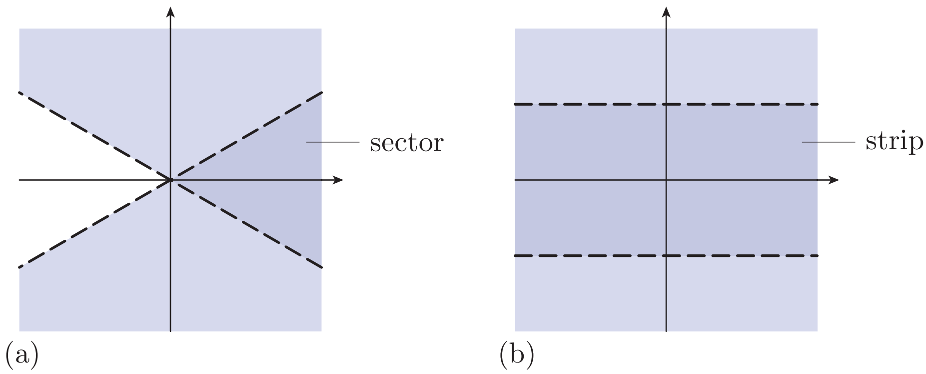


Figure 4.21 (a) Two half-planes intersecting in a sector (b) Two half-planes intersecting in a horizontal strip

A significant difference between the left and right regions illustrated in Figure 4.21 is that the two boundary extended lines in Figure 4.21(a) (the dashed lines) intersect at exactly two points in $\widehat{\mathbb{C}}$, namely 0 and ∞, whereas the two boundary extended lines in Figure 4.21(b) intersect at only one point, ∞. Regions of the former type are called *lunes*.

Definitions

A **lune** is a set in $\widehat{\mathbb{C}}$ formed from the intersection of two generalised open discs whose boundaries in $\widehat{\mathbb{C}}$, which are generalised circles, intersect at exactly two points.

The two intersection points are called the **vertices** of the lune.

For example, four lunes are displayed in Figure 4.22. The first is the intersection of two open discs, the second is the intersection of the inside of the left circle and the outside of the right circle, the third is the intersection of the outside of the left circle and the outside of the right circle (it includes the point ∞), and the fourth is the intersection of the outside of the circle and a half-plane.

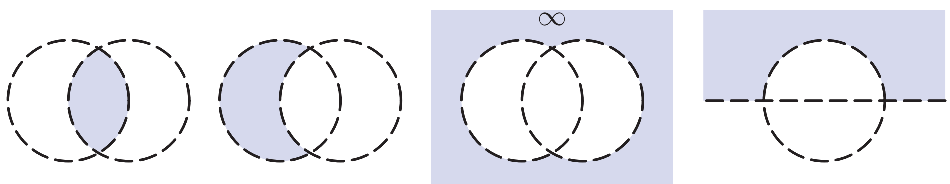


Figure 4.22 Four examples of lunes (represented by the shaded areas)

The second of these sets resembles a crescent-shaped moon, and it is for this reason that these sets are described as 'lunes'.

All lunes are open and connected, so those that do not contain the point ∞ (such as the first, second and fourth lunes in Figure 4.22) are regions in the complex plane $\mathbb{C}$.

Next we define the *angle* of a lune $\mathcal{R}$. Before continuing, you may wish to refer to Subsection 4.3 of Unit A4 for a discussion of the angle from one smooth path to another.

Let D_1 and D_2 be generalised open discs such that $\mathcal{R} = D_1 \cap D_2$ is a lune. Let C_1 and C_2 be smooth paths that traverse the boundaries in $\mathbb{C}$ of D_1 and D_2, respectively. We choose the directions of C_1 and C_2 such that D_1 lies to the *left* of C_1, and D_2 lies to the *right* of C_2; see Figure 4.23.

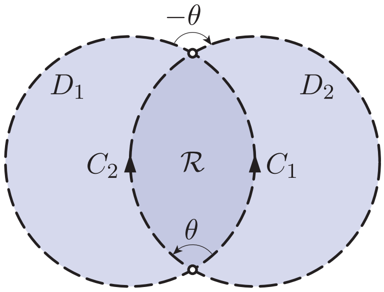


Figure 4.23 The angle θ of a lune

The **angle** of the lune $\mathcal{R}$ is the absolute value of the angle from C_1 to C_2 at a vertex (not ∞) of $\mathcal{R}$, as illustrated in Figure 4.23. We need to take the absolute value because the angle from C_1 to C_2 at one vertex is the negative of the angle from C_1 to C_2 at the other vertex. The angles of various lunes are shown in Figure 4.24.

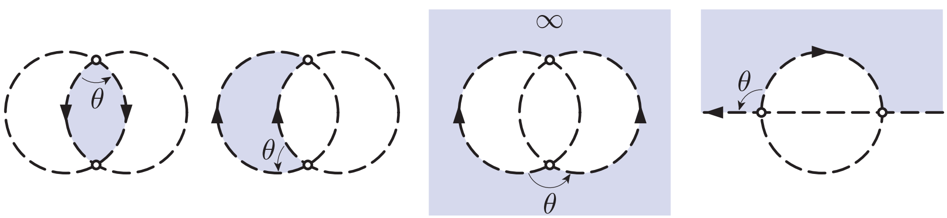


Figure 4.24 Angles of various lunes

As you know, Möbius transformations map generalised circles onto generalised circles, and they are conformal mappings, so they preserve angles. These two facts can be used to construct a conformal mapping between two lunes of the same angle, as the following example demonstrates.

In this example we use the convention from Unit A1 of writing the set $\{z : z \in A \text{ and } z \in B\}$ in the abbreviated form $\{z : z \in A,\ z \in B\}$.

Example 4.7

Find a one-to-one conformal mapping from the open half-disc

$$\mathcal{R} = \{z : |z| < 1,\ \operatorname{Im} z > 0\}$$

onto the open sector

$$\mathcal{S} = \{w : \operatorname{Re} w > 0,\ \operatorname{Im} w > 0\}.$$

The regions are illustrated in Figure 4.25.

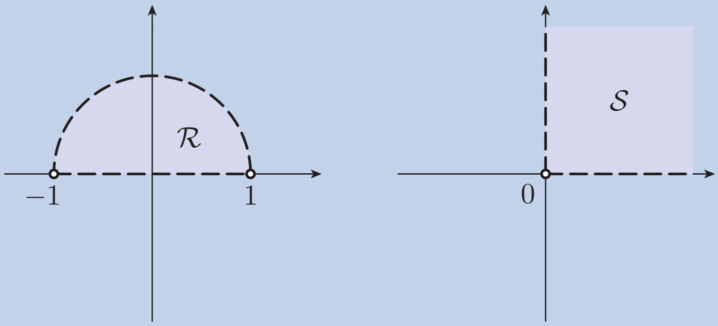


Figure 4.25 The open half-disc $\mathcal{R} = \{z : |z| < 1,\ \operatorname{Im} z > 0\}$ and the open sector $\mathcal{S} = \{w : \operatorname{Re} w > 0,\ \operatorname{Im} w > 0\}$

Solution

Both regions are lunes. We label the boundary arcs Γ_1, Γ_2, Γ_1' and Γ_2', as shown in Figure 4.26, where Γ_1 is the line segment from -1 to 1, Γ_1' is the positive real axis traversed from 0 to ∞, and so on.

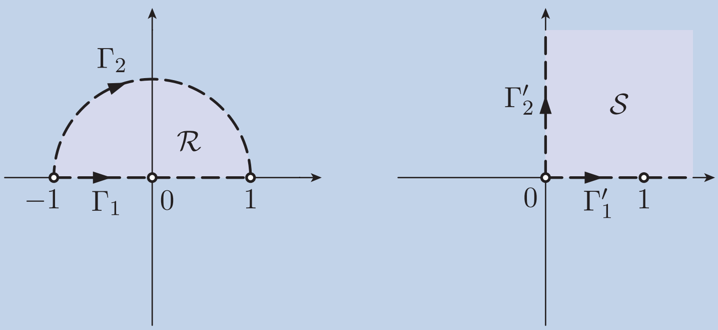


Figure 4.26 Boundary arcs Γ_1 and Γ_2 of $\mathcal{R}$ and Γ_1' and Γ_2' of $\mathcal{S}$

The angle of both lunes is $\pi/2$, so there should be a Möbius transformation f that maps $\mathcal{R}$ onto $\mathcal{S}$. This transformation f must map the vertices -1 and 1 of $\mathcal{R}$ to the vertices 0 and ∞ of $\mathcal{S}$, in some order. We choose the order to match the direction of the paths, in the sense that the initial point of Γ_1 is sent to the initial point of Γ_1', and the final point of Γ_1 is sent to the final point of Γ_1'. That is, we define $f(-1) = 0$ and $f(1) = \infty$.

Let us also choose f to map some other point on the boundary of $\mathcal{R}$ to some other point on the boundary of $\mathcal{S}$, just as we did in earlier examples, making sure that either the two points lie on Γ_1 and Γ_1', or else they lie on Γ_2 and Γ_2'. A convenient choice is 0 on Γ_1 and 1 on Γ_1', so $f(0) = 1$.

To sum up, we choose f to satisfy

$$f(-1) = 0, \quad f(1) = \infty \quad \text{and} \quad f(0) = 1.$$

By the Explicit Formula for Möbius Transformations, we have

$$f(z) = \frac{(z+1)}{(z-1)}\frac{(0-1)}{(0+1)} = \frac{z+1}{-z+1}.$$

This transformation maps Γ_1 to an arc of a generalised circle that passes through the point 1 and has initial point 0 and final point ∞. Hence $f(\Gamma_1) = \Gamma_1'$. Next, since f is conformal at -1, the angle from $f(\Gamma_1)$ to $f(\Gamma_2)$ at $f(-1) = 0$ is $\pi/2$. Since $f(\Gamma_2)$ is an arc of a generalised circle from 0 to ∞, we see that $f(\Gamma_2) = \Gamma_2'$.

We have now seen that f maps the boundary $\partial\mathcal{R}$ in $\widehat{\mathbb{C}}$ of $\mathcal{R}$ onto the boundary $\partial\mathcal{S}$ in $\widehat{\mathbb{C}}$ of $\mathcal{S}$. It follows that $f(\mathcal{R}) = \mathcal{S}$, since $\mathcal{R}$ lies to the left of Γ_1 and $\mathcal{S}$ lies to the left of Γ_1'.

Finally, because f is one-to-one and conformal throughout $\widehat{\mathbb{C}}$, we see that f is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

The statement $f(\mathcal{R}) = \mathcal{S}$ in Example 4.7 can be justified by an argument similar to that used to prove Theorem 4.1, as follows.

The complement in $\widehat{\mathbb{C}}$ of $\partial\mathcal{R}$ has two connected parts,

$$\mathcal{R} \quad \text{and} \quad \mathcal{R}' = \{z : |z| > 1 \text{ or } \operatorname{Im} z < 0\} \cup \{\infty\},$$

and the complement in $\widehat{\mathbb{C}}$ of $\partial\mathcal{S}$ has two connected parts,

$$\mathcal{S} \quad \text{and} \quad \mathcal{S}' = \{w : \operatorname{Re} w < 0 \text{ or } \operatorname{Im} w < 0\}$$

(see Figure 4.27). Since f is a one-to-one mapping from $\widehat{\mathbb{C}}$ onto itself, and $\widehat{\mathbb{C}} = \mathcal{R} \cup \partial\mathcal{R} \cup \mathcal{R}'$, we see that

$$\widehat{\mathbb{C}} = f(\mathcal{R}) \cup f(\partial\mathcal{R}) \cup f(\mathcal{R}') = f(\mathcal{R}) \cup \partial\mathcal{S} \cup f(\mathcal{R}').$$

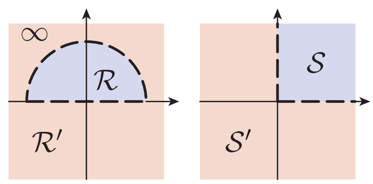


Figure 4.27 The sets $\mathcal{R}$, $\mathcal{R}'$, $\mathcal{S}$ and $\mathcal{S}'$

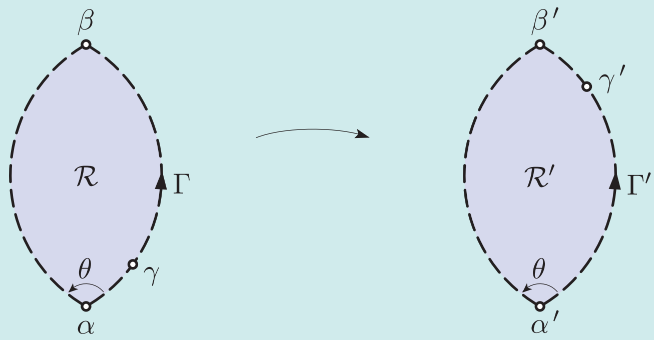
The sets $f(\mathcal{R})$ and $f(\mathcal{R}')$ are connected, since they are the continuous images of connected sets, so one of them must be $\mathcal{S}$ and the other $\mathcal{S}'$. Now, $\mathcal{R}$ lies to the left of Γ_1, and $\mathcal{S}$ lies to the left of Γ_1', so we deduce that $f(\mathcal{R}) = \mathcal{S}$.

The steps of Example 4.7 are summarised in the following strategy for finding conformal mappings between lunes.

Strategy for mapping lunes

Let $\mathcal{R}$ be a lune with vertices α and β, and let Γ be a smooth path in the complex plane that traverses one of the boundary arcs of $\mathcal{R}$ from α to β, excluding the endpoints, such that $\mathcal{R}$ lies to the *left* of Γ. Let $\mathcal{R}'$, α', β' and Γ' be defined in a similar way.

(Alternatively we can replace 'left' by 'right' for *both* paths Γ and Γ'.)



Suppose that the angles of $\mathcal{R}$ and $\mathcal{R}'$ are equal.

To find a Möbius transformation f that maps $\mathcal{R}$ onto $\mathcal{R}'$, carry out the following steps.

1. Define $f(\alpha) = \alpha'$ and $f(\beta) = \beta'$.
2. Choose any points γ on Γ and γ' on Γ', and define $f(\gamma) = \gamma'$.
3. Use the Implicit or Explicit Formula for Möbius Transformations to determine f.

In this strategy the notation Γ' is used for the image of Γ. Despite using the dash notation, Γ' has nothing to do with derivatives. Also, the point γ chosen on Γ has nothing to do with a parametrisation of Γ (it is a point, not a parametrisation).

We can reason in a similar way to Example 4.7 to see that the Möbius transformation f constructed in the strategy does indeed map $\mathcal{R}$ onto $\mathcal{S}$. In brief, since f maps α to α', β to β' and γ to γ', it must map the arc Γ onto Γ'. Then the other arc on the boundary of $\mathcal{R}$ between α and β maps onto the other arc on the boundary of $\mathcal{R}'$ between α' and β', because f preserves the angle θ of the lune. We have now seen that f maps the boundary of $\mathcal{R}$ to the boundary of $\mathcal{S}$, and an argument similar to that in the proof of Theorem 4.1 can be applied to see that f maps $\mathcal{R}$ onto $\mathcal{S}$.

Try using the strategy to find a conformal mapping between lunes in the following exercise.

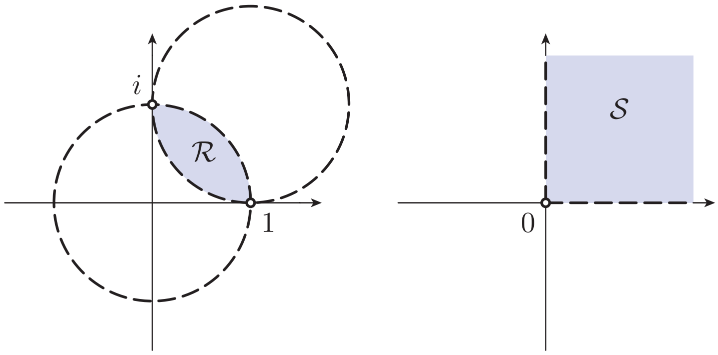
Exercise 4.8

Find a one-to-one conformal mapping from the region

$$\mathcal{R} = \{z : |z| < 1,\ |z - (1+i)| < 1\}$$

onto the open sector

$$\mathcal{S} = \{w : \operatorname{Re} w > 0,\ \operatorname{Im} w > 0\}.$$



4.3 Composing conformal mappings

This subsection continues our study of conformal mappings between regions. We will find that by composing the basic conformal mappings of the previous subsection, we can construct conformal mappings between a wide variety of regions.

Let us start with an example. Suppose that we wish to find a one-to-one conformal mapping from the upper-right quadrant $\mathcal{R}$ of the complex plane onto the open unit disc $\mathcal{S}$, as shown in Figure 4.28.

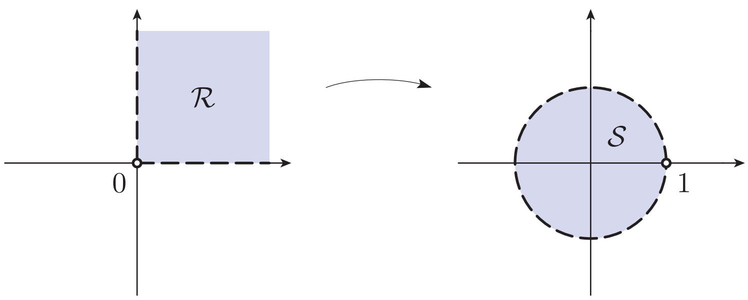


Figure 4.28 The upper-right quadrant $\mathcal{R}$ and the open unit disc $\mathcal{S}$

None of the basic conformal mappings that we have considered so far will do this directly, but we can split the problem into two stages by introducing an intermediate region, the upper half-plane $\mathcal{R}_1$, as shown in Figure 4.29. We have labelled the plane in which this intermediate region lies the z_1-plane, because we will denote a general point in that plane by z_1.

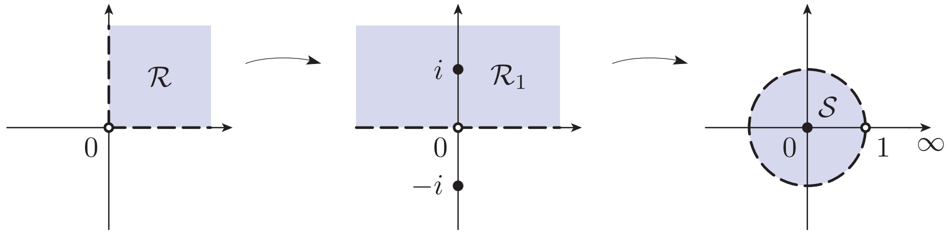


Figure 4.29 The upper-right quadrant $\mathcal{R}$, the upper half-plane $\mathcal{R}_1$ and the open unit disc $\mathcal{S}$

Our task now is to find a mapping from $\mathcal{R}$ onto $\mathcal{R}_1$, and then find a mapping from $\mathcal{R}_1$ onto $\mathcal{S}$. For the first mapping, reasoning in a similar way to Example 4.6, we see that the function

$$z_1 = z^2$$

squares moduli and doubles arguments, so it is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{R}_1$.

Next we can find a Möbius transformation that maps $\mathcal{R}_1$ onto $\mathcal{S}$ by using the inverse points method demonstrated in Example 4.5. We choose inverse points i and $-i$ with respect to the boundary of $\mathcal{R}_1$, and the point 0 on this boundary. And we choose inverse points 0 and ∞ with respect to the boundary of $\mathcal{S}$, and the point 1 on this boundary. By the Explicit Formula for Möbius Transformations, the unique Möbius transformation that sends i to 0, $-i$ to ∞ and 0 to 1 is

$$w = \frac{-z_1 + i}{z_1 + i};$$

this is a one-to-one conformal mapping from $\mathcal{R}_1$ onto $\mathcal{S}$. (There are other ways of obtaining a mapping from $\mathcal{R}_1$ onto $\mathcal{S}$, of course; for example, we could have first applied the rotation $z_1 \longmapsto -iz_1$ to map $\mathcal{R}_1$ onto the right half-plane, and then used one of the mappings from Table 4.1.)

We now compose this Möbius transformation with the square function to obtain a mapping f from the quadrant $\mathcal{R}$ onto the disc $\mathcal{S}$ given by

$$f(z) = w = \frac{-z_1 + i}{z_1 + i} = \frac{-z^2 + i}{z^2 + i}.$$

We know that f is one-to-one and conformal on $\mathcal{R}$ because f is the composition of the square function, which is one-to-one and conformal on $\mathcal{R}$, and a Möbius transformation, which is one-to-one and conformal on $\mathcal{R}_1$.

Let us now find the inverse function of f, which maps the open unit disc $\mathcal{S}$ back onto the quadrant $\mathcal{R}$. You may be tempted to find the inverse function by taking the equation for f and solving for z in terms of w. Unfortunately, this process will involve a square root, and it is not immediately obvious *which* square root should be used. For this reason we build up the inverse function of f from the inverse functions of its constituent mappings.

It is simple enough to map the disc $\mathcal{S}$ back onto the half-plane $\mathcal{R}_1$: we just use the inverse function of the Möbius transformation, which is given by

$$z_1 = \frac{iw - i}{-w - 1} = \frac{-iw + i}{w + 1}.$$

A little more care is needed with the square function to ensure that the correct square root is used. To do this, we note that the quadrant $\mathcal{R}$ lies in the image set of the principal square root function. So the principal square root $z = \sqrt{z_1}$ is the square root to use. Consequently, the inverse function of f is

$$f^{-1}(w) = \sqrt{z_1} = \sqrt{\frac{-iw + i}{w + 1}},$$

which is a one-to-one conformal mapping from $\mathcal{S}$ onto $\mathcal{R}$. The mapping f and its inverse function are illustrated in Figure 4.30.

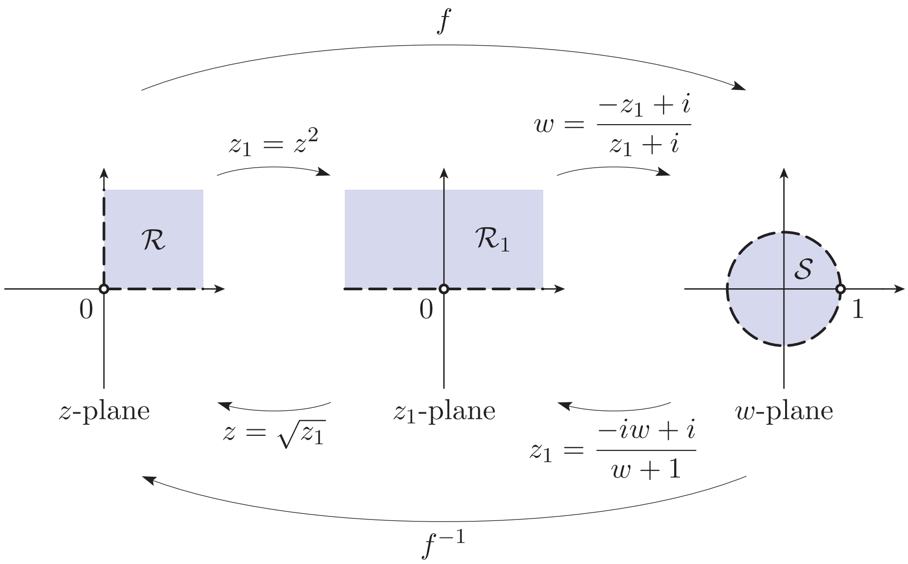


Figure 4.30 Conformal mapping from the upper-right quadrant $\mathcal{R}$ onto the open unit disc $\mathcal{S}$

Here is another example.

Example 4.8

Find a one-to-one conformal mapping f from the region

$$\mathcal{R} = \{z : |z| < 1,\ \operatorname{Im} z > 0\},$$

shown in Figure 4.31, onto the upper half-plane $\mathcal{S} = \{z : \operatorname{Im} z > 0\}$. Determine the rule for the inverse function f^{-1}.

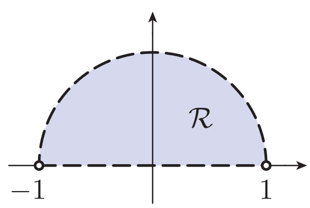


Figure 4.31 The region $\mathcal{R} = \{z : |z| < 1,\ \operatorname{Im} z > 0\}$

Solution

To find the required mapping we first introduce a convenient intermediate region $\mathcal{R}_1$, as shown in Figure 4.32.

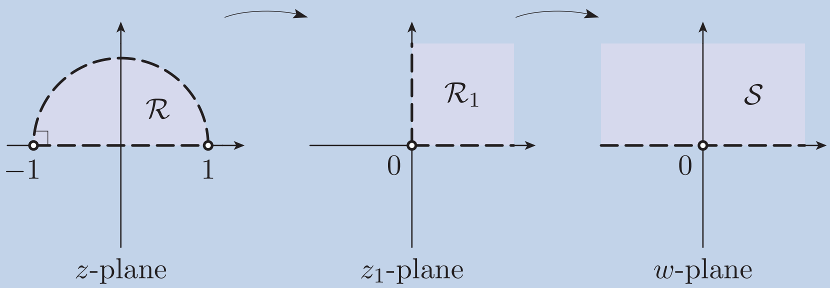


Figure 4.32 The region $\mathcal{R} = \{z : |z| < 1,\ \operatorname{Im} z > 0\}$, the upper-right quadrant $\mathcal{R}_1$ and the upper half-plane $\mathcal{S}$

We will find one-to-one conformal mappings from $\mathcal{R}$ onto $\mathcal{R}_1$ and from $\mathcal{R}_1$ onto $\mathcal{S}$, and then compose these two mappings to obtain a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

The regions $\mathcal{R}$ and $\mathcal{R}_1$ are both lunes of angle $\pi/2$, so we can apply the strategy for mapping lunes to find a Möbius transformation that maps one lune onto the other. In fact, we have considered these two particular regions already in Example 4.7, where we saw that the Möbius transformation

$$z_1 = \frac{z+1}{-z+1}$$

is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{R}_1$.

Next we need to map the sector $\mathcal{R}_1$ onto the half-plane $\mathcal{S}$. As we saw in Table 4.1, a sector can be mapped onto a half-plane using a power function. In this case the function

$$w = z_1^2$$

squares the modulus of each complex number and doubles the argument. Therefore it is a one-to-one mapping from

$$\mathcal{R}_1 = \{z_1 : 0 < \operatorname{Arg} z_1 < \pi/2\} \quad \text{onto} \quad \mathcal{S} = \{w : 0 < \operatorname{Arg} w < \pi\}.$$

This function is analytic throughout $\mathbb{C}$, so it is a one-to-one conformal mapping from $\mathcal{R}_1$ onto $\mathcal{S}$.

The composition of these two functions is a one-to-one conformal mapping f from $\mathcal{R}$ onto $\mathcal{S}$ given by

$$f(z) = \left(\frac{z+1}{-z+1}\right)^2.$$

To find the rule for the inverse function f^{-1}, we need to find the inverse functions of each of the constituent mappings.

By Theorem 2.3, the inverse function of the Möbius transformation is

$$z = \frac{z_1 - 1}{z_1 + 1}.$$

The inverse function of the power function $w = z_1^2$ is the principal square root function $z_1 = \sqrt{w}$, since this maps the upper half-plane $\mathcal{S}$ onto the first quadrant $\mathcal{R}_1$. Hence

$$f^{-1}(w) = \frac{\sqrt{w} - 1}{\sqrt{w} + 1}.$$

Exercise 4.9

Find a one-to-one conformal mapping f from the cut plane

$$\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\},$$

illustrated in Figure 4.33, onto the open unit disc

$$\mathcal{S} = \{w : |w| < 1\}.$$

Determine the rule for the inverse function f^{-1}.

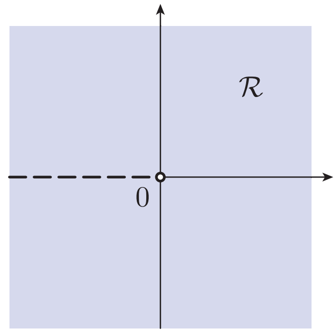


Figure 4.33 The cut plane $\mathcal{R} = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$

We finish this subsection by proving a result about the tan function and its inverse function, which was promised in Subsection 3.2 of Unit C2. We also state (but do not prove) a similar result for $\sin^{-1}$. The proof demonstrates the usefulness of composing conformal mappings; however, it is quite long, so it may be omitted on a first reading.

Theorem 4.2

The function tan is a one-to-one conformal mapping from

$$\mathcal{R} = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\} \text{ onto } \mathcal{S} = \mathbb{C} - \{iv : v \in \mathbb{R},\ |v| \geq 1\}$$

with inverse function

$$\tan^{-1} w = \frac{1}{2i} \operatorname{Log}\left(\frac{1 + iw}{1 - iw}\right).$$

Theorem 4.2 is illustrated in Figure 4.34.

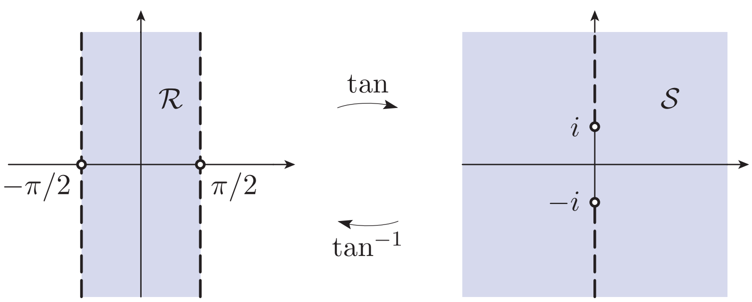


Figure 4.34 The conformal mapping tan and its inverse function

Proof Observe that $\tan z$ can be written in the form

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\left(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right) = \frac{e^{2iz} - 1}{ie^{2iz} + i}.$$

This suggests a method for expressing tan as the composition of the basic conformal mappings

$$z_1 = 2iz, \quad z_2 = e^{z_1} \quad \text{and} \quad w = \frac{z_2 - 1}{iz_2 + i},$$

in that order. The effect of these basic conformal mappings is shown in Figure 4.35.

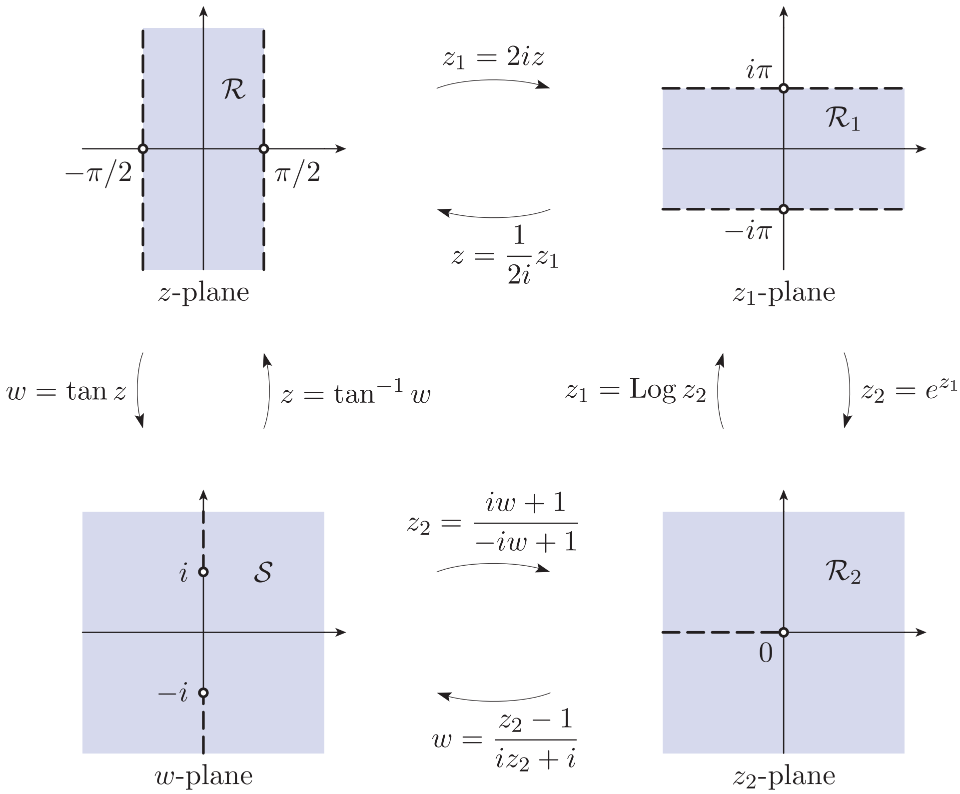


Figure 4.35 Expressing tan as a composition of basic conformal mappings

The linear function $z_1 = 2iz$ rotates the vertical strip $\mathcal{R}$ anticlockwise through an angle of $\pi/2$ about the origin and then scales it by a factor of 2. This gives the horizontal strip $\mathcal{R}_1$ shown at the top right of Figure 4.35. Next, using the methods of Subsection 4.2 on mapping horizontal strips under the exponential function, we see that the function $z_2 = e^{z_1}$ maps $\mathcal{R}_1$ onto the cut plane $\mathcal{R}_2 = \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$. So all that remains is to examine the effect that the Möbius transformation $w = (z_2 - 1)/(iz_2 + i)$ has on this cut plane.

Since the Möbius transformation is a one-to-one mapping of $\widehat{\mathbb{C}}$ to itself, the easiest way to find the image of the cut plane is to first find the image of the cut $L = \{t \in \mathbb{R} : t \leq 0\} \cup \{\infty\}$. We observe that the transformation sends 0 to i, -1 to ∞ and ∞ to $-i$. Consequently, the image of L is an arc of the generalised circle that passes from i to ∞ to $-i$. This arc looks like two separate half-lines (on the imaginary axis) in the complex plane, but in the extended complex plane they are connected by the point at infinity. The region $\mathcal{R}_2$ is mapped to $\mathcal{S}$, which is the complement in $\mathbb{C}$ of these two line segments.

By composing these one-to-one conformal mappings, we deduce that tan is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

The formula for the inverse function can be obtained by finding the inverse functions of the basic constituent mappings, which are

$$z_2 = \frac{iw+1}{-iw+1}, \quad z_1 = \operatorname{Log} z_2 \quad \text{and} \quad z = \frac{1}{2i} z_1.$$

Composing these, we see that

$$\tan^{-1} w = \frac{1}{2i} \operatorname{Log}\left(\frac{1+iw}{1-iw}\right), \quad \text{for } w \in \mathcal{S},$$

as required. ■

Next we state a theorem about the inverse sin function. This theorem can be proved using similar techniques to those used to prove Theorem 4.2; we omit the details.

Theorem 4.3

The function sin is a one-to-one conformal mapping from

$$\mathcal{R} = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\} \text{ onto } \mathcal{S} = \mathbb{C} - \{u \in \mathbb{R} : |u| \geq 1\}$$

with inverse function

$$\sin^{-1} w = \frac{1}{i} \operatorname{Log}\left(iw + \sqrt{1-w^2}\right).$$

Theorem 4.3 is illustrated in Figure 4.36.

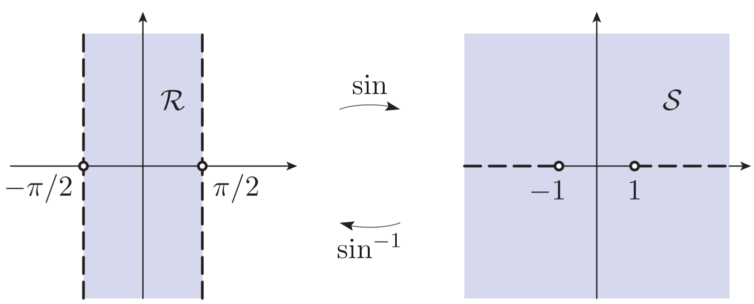


Figure 4.36 The conformal mapping sin and its inverse function

Further exercises

Exercise 4.10

Write down the boundary in $\widehat{\mathbb{C}}$ of each of the following subsets of $\widehat{\mathbb{C}}$.

(a) $\{z : |z - i| < 2\}$ (b) $\{z : \operatorname{Im} z < -1\}$

(c) $\{z : |z + i| > 1\} \cup \{\infty\}$ (d) $\{z : |z + i| > 1\}$

Exercise 4.11

(a) Find the image of the open disc $D = \{z : |z - 1| < 2\}$ under the Möbius transformation

$$f(z) = \frac{z + i}{z + 1}.$$

(b) Find the image of the open disc $D = \{z : |z + i| < 1\}$ under the Möbius transformation

$$f(z) = \frac{z - i}{z + i}.$$

Hence find the image of the punctured open disc $\mathcal{R} = \{z : 0 < |z + i| < 1\}$ under f.

Exercise 4.12

Find a Möbius transformation that maps the open half-plane $\{z : \operatorname{Re} z + \operatorname{Im} z < 1\}$ onto the open disc $\{z : |z| < 2\}$.

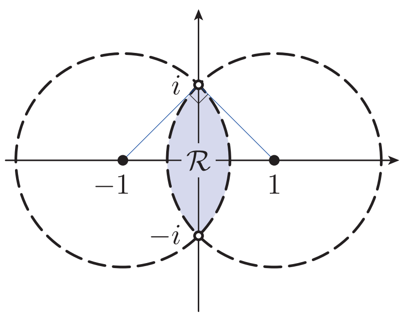


Figure 4.37 The lune $\mathcal{R} = \{z : |z-1| < \sqrt{2},\ |z+1| < \sqrt{2}\}$

Exercise 4.13

Find a one-to-one conformal mapping f from the lune

$$\mathcal{R} = \{z : |z - 1| < \sqrt{2},\ |z + 1| < \sqrt{2}\},$$

illustrated in Figure 4.37, onto the right half-plane $\mathcal{S} = \{z : \operatorname{Re} z > 0\}$. Determine the rule for the inverse function f^{-1}.

5 The Riemann Mapping Theorem

After working through this section, you should be able to:

- appreciate the role of the Riemann Mapping Theorem in complex analysis.

This section is intended for reading only (it will not be assessed).

In Section 4 you saw how to construct one-to-one conformal mappings between many pairs of regions. For example, from Exercise 4.9, the function

$$f(z) = \frac{\sqrt{z}-1}{\sqrt{z}+1}$$

is a one-to-one conformal mapping from the cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ onto the open unit disc $\{w : |w| < 1\}$ (see Figure 5.1).

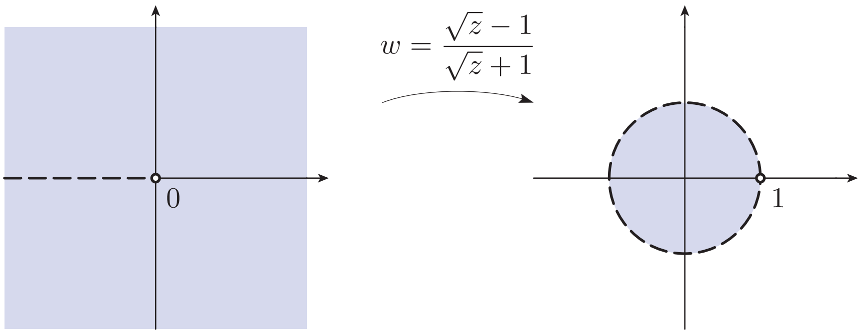


Figure 5.1 Conformal mapping from the cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ onto the open unit disc

All the conformal mappings constructed in Section 4 were given by explicit formulas involving elementary analytic functions. In this section we consider a more general problem, tackled by Riemann in his dissertation, which was presented to the University of Göttingen in 1851:

> For which pairs of simply connected regions $\mathcal{R}_1$ and $\mathcal{R}_2$ is there a one-to-one conformal mapping from $\mathcal{R}_1$ onto $\mathcal{R}_2$?

Recall that a region is simply connected if it has no 'holes' in it (Subsection 1.1 of Unit B2).

It is easy to find a pair of simply connected regions for which such a mapping does not exist. For example, if $\mathcal{R}_1 = \mathbb{C}$ and $\mathcal{R}_2 = \{w : |w| < 1\}$, then there is no one-to-one conformal mapping from $\mathcal{R}_1$ onto $\mathcal{R}_2$. For any such mapping f would be a bounded entire function and hence would have to be constant by Liouville's Theorem (Theorem 2.2 of Unit B2).

In view of this example, it is perhaps surprising that Riemann came to the following conclusion:

> If $\mathcal{R}_1$ and $\mathcal{R}_2$ are *any* two simply connected regions, neither equal to $\mathbb{C}$, then there is a one-to-one conformal mapping from $\mathcal{R}_1$ onto $\mathcal{R}_2$.

Since simply connected regions can be complicated, we should not expect to be able to construct such a mapping explicitly. For example, see the complicated simply connected region of Figure 5.2, which is repeated from Figure 4.20 of Unit A3. This represents a rectangle with infinitely many vertical line segments removed, accumulating on the right-hand side. For clarity, and contrary to our usual conventions, we have drawn the boundary of the region with solid lines rather than dashed lines.

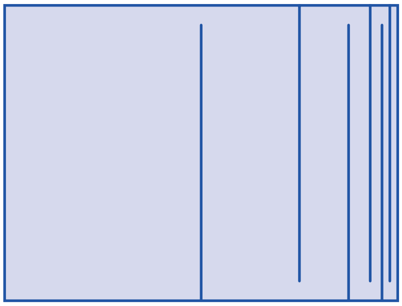


Figure 5.2 A simply connected region formed by removing infinitely many vertical line segments from a rectangle

However, Riemann did not claim to be able to construct the mappings: he claimed only their *existence*, without specifying how to calculate them. The following special case of his assertion, in which one of the regions is the open unit disc, is known as the Riemann Mapping Theorem.

Theorem 5.1 Riemann Mapping Theorem

There is a one-to-one conformal mapping from any simply connected region other than $\mathbb{C}$ onto the open unit disc.

In fact, if α is a point in a simply connected region $\mathcal{R}$ (other than $\mathbb{C}$), then it can be shown that there is a unique one-to-one conformal mapping from $\mathcal{R}$ onto the open unit disc D such that

$$f(\alpha) = 0 \quad \text{and} \quad f'(\alpha) \text{ is real and positive.}$$

Theorem 5.1 leads to a complete solution to Riemann's problem, as follows. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are simply connected regions, neither equal to $\mathbb{C}$, then Theorem 5.1 implies the existence of one-to-one conformal mappings f_1 and f_2 from $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, onto D (see Figure 5.3). Then f_2^{-1} is a one-to-one conformal mapping from D onto $\mathcal{R}_2$, and hence the composite function $f = f_2^{-1} \circ f_1$ is a one-to-one conformal mapping from $\mathcal{R}_1$ onto $\mathcal{R}_2$.

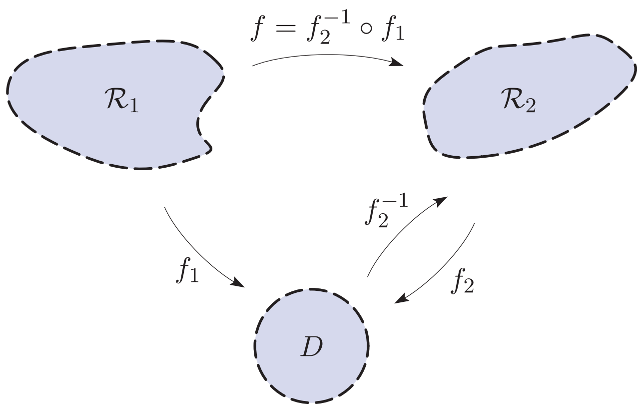


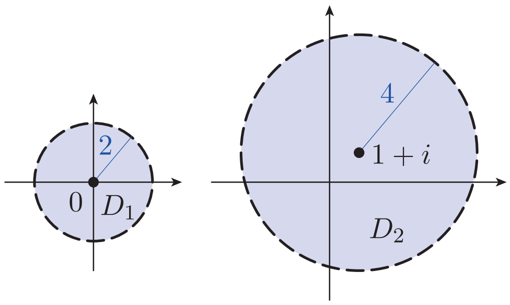
Figure 5.3 Conformal mappings between simply connected regions $\mathcal{R}_1$, $\mathcal{R}_2$ and the open unit disc D

Riemann did not supply a complete proof of Theorem 5.1, and his argument can be made to work only for regions with well-behaved boundaries (better behaved than that of Figure 5.2). A complete proof of Theorem 5.1 was not given until the early 1900s by the combined efforts of many complex analysts. It would take too long to supply all the details of a proof here, but, in brief, one method is to construct a sequence of functions (f_n) that map $\mathcal{R}$ one-to-one and conformally onto regions that approximate D more and more closely as $n \to \infty$, and then show that the limit function $f = \lim_{n\to\infty} f_n$ maps $\mathcal{R}$ one-to-one and conformally onto D.

In his dissertation, Riemann in fact made a stronger assertion than the Riemann Mapping Theorem. He stated that if $\mathcal{R}_1$ and $\mathcal{R}_2$ are two simply connected regions, neither equal to $\mathbb{C}$, then there is a one-to-one *continuous* function f from $\mathcal{R}_1 \cup \partial\mathcal{R}_1$ onto $\mathcal{R}_2 \cup \partial\mathcal{R}_2$ such that the restriction of f to $\mathcal{R}_1$ is a one-to-one conformal mapping from $\mathcal{R}_1$ onto $\mathcal{R}_2$. (Here $\partial\mathcal{R}_1$ and $\partial\mathcal{R}_2$ are the boundaries of $\mathcal{R}_1$ and $\mathcal{R}_2$ in $\widehat{\mathbb{C}}$.) This was later proved to be true for simply connected regions with well-behaved boundaries by the Greek mathematician Constantin Carathéodory (whom you met in Unit C2) in 1913. However, in general the assertion is false; for example, it fails when $\mathcal{R}_1$ is the unit disc and $\mathcal{R}_2$ is the region shown in Figure 5.2.

Solutions to exercises

Solution to Exercise 1.1



We can map D_1 onto D_2 in two stages. We first scale by the factor 2 and then translate by $1+i$; no rotation is needed. Overall, this can be achieved by the linear function

$$f(z) = 2z + (1+i).$$

Solution to Exercise 1.2

By substituting

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

in the equation $y + 4x = 1$, we obtain

$$\left(\frac{-v}{u^2+v^2}\right) + 4\left(\frac{u}{u^2+v^2}\right) = 1.$$

Multiplying through by u^2+v^2 gives

$$u^2 + v^2 - 4u + v = 0.$$

If we complete the squares, then this equation becomes

$$(u-2)^2 + \left(v+\tfrac{1}{2}\right)^2 = \tfrac{17}{4}.$$

(The image of the line is the circle centred at $2 - \frac{1}{2}i$ of radius $\frac{1}{2}\sqrt{17}$, except for 0.)

Solution to Exercise 1.3

By substituting

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

in the equation $y = x$, we obtain

$$\frac{-v}{u^2+v^2} = \frac{u}{u^2+v^2},$$

that is, $v = -u$, for $u^2+v^2 \neq 0$.

(The image is the reflection of the given line in the real axis.)

Solution to Exercise 1.4

(a) On replacing x by $u/\left(u^2+v^2\right)$ and y by $-v/\left(u^2+v^2\right)$ in $x^2+y^2=4$, we obtain

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 = 4.$$

This equation simplifies to give

$$\frac{1}{u^2+v^2} = 4,$$

that is,

$$u^2+v^2 = \tfrac{1}{4},$$

which is an equation for the circle of radius $\frac{1}{2}$ centred at the origin.

(b) The equation $(x-3)^2 + (y-4)^2 = 25$ can be written as

$$x^2 + y^2 - 6x - 8y = 0.$$

On replacing x by $u/\left(u^2+v^2\right)$ and y by $-v/\left(u^2+v^2\right)$, for $x+iy \neq 0$, we obtain

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 6\left(\frac{u}{u^2+v^2}\right) - 8\left(\frac{-v}{u^2+v^2}\right) = 0.$$

The first two terms combine to give

$$\frac{1}{u^2+v^2},$$

so the equation becomes

$$\frac{1}{u^2+v^2} - 6\left(\frac{u}{u^2+v^2}\right) - 8\left(\frac{-v}{u^2+v^2}\right) = 0.$$

By multiplying through by u^2+v^2 and rearranging, we obtain

$$6u - 8v = 1,$$

which is the equation of a line.

Solution to Exercise 1.5

By Theorem 1.3, the reciprocal function maps the line or circle

$$a(x^2+y^2)+bx+cy+d=0$$

(excluding the origin) onto the line or circle

$$d(u^2+v^2)+bu-cv+a=0$$

(again excluding the origin), where $a,b,c,d\in\mathbb{R}$ and $b^2+c^2>4ad$. By Theorem 1.2, the first equation represents a line if and only if $a=0$, and it represents a line or circle that passes through the origin if and only if $d=0$. For the second equation, the roles of a and d are reversed, so the second equation represents a line if and only if $d=0$, and it represents a line or circle that passes through the origin if and only if $a=0$.

The four cases then arise as follows.

(a) $a=d=0$

(b) $a=0,\ d\neq 0$

(c) $a\neq 0,\ d=0$

(d) $a\neq 0,\ d\neq 0$

Solution to Exercise 1.6

Using Theorem 1.2 and Exercise 1.5, we obtain the following classification.

(a) This is a line that does not pass through the origin, so its image is a circle through the origin.

(b) This is a circle through the origin, so its image is a line that does not pass through the origin.

(c) This is a circle that does not pass through the origin, so its image is a circle that does not pass through the origin.

(d) This is a line through the origin, so its image is a line through the origin.

Solution to Exercise 1.7

(a) The function f has a single singularity in $\mathbb{C}$, at $-1/3$. Since $f(z)\to\infty$ as $z\to -1/3$, we define $f(-1/3)=\infty$. Next,

$$f(z)=\frac{6z+4}{3z+1}=\frac{6+4/z}{3+1/z}.$$

Therefore

$$f(z)\to\frac{6}{3}=2 \text{ as } z\to\infty,$$

so we define $f(\infty)=2$. Hence the extended function is

$$f(z)=\begin{cases}\dfrac{6z+4}{3z+1}, & z\in\mathbb{C}-\{-1/3\},\\ \infty, & z=-1/3,\\ 2, & z=\infty.\end{cases}$$

(b) The function f has a single singularity in $\mathbb{C}$, at 0. Since $f(z)\to\infty$ as $z\to 0$, we define $f(0)=\infty$. Next,

$$f(z)=\frac{1}{z}\to 0 \text{ as } z\to\infty,$$

so we define $f(\infty)=0$. Hence the extended function is

$$f(z)=\begin{cases}\dfrac{1}{z}, & z\in\mathbb{C}-\{0\},\\ \infty, & z=0,\\ 0, & z=\infty.\end{cases}$$

(c) Here f is entire and

$$f(z)=5z+7\to\infty \text{ as } z\to\infty,$$

so we define $f(\infty)=\infty$. Hence the extended function is

$$f(z)=\begin{cases}5z+7, & z\in\mathbb{C},\\ \infty, & z=\infty.\end{cases}$$

(d) The function f has a single singularity in $\mathbb{C}$, at $-1/2$. Since $f(z)\to\infty$ as $z\to -1/2$, we define $f(-1/2)=\infty$. Next,

$$f(z)=\frac{1}{2z+1}=\frac{1/z}{2+1/z}.$$

Therefore

$$f(z)\to\frac{0}{2}=0 \text{ as } z\to\infty,$$

so we define $f(\infty)=0$. Hence the extended function is

$$f(z)=\begin{cases}\dfrac{1}{2z+1}, & z\in\mathbb{C}-\{-1/2\},\\ \infty, & z=-1/2,\\ 0, & z=\infty.\end{cases}$$

(e) The function f has a single singularity in $\mathbb{C}$, at 0. Since $f(z) \to \infty$ as $z \to 0$, we define $f(0) = \infty$. Next,

$$f(z) = \frac{z^2+1}{z} = z + \frac{1}{z} \to \infty \text{ as } z \to \infty,$$

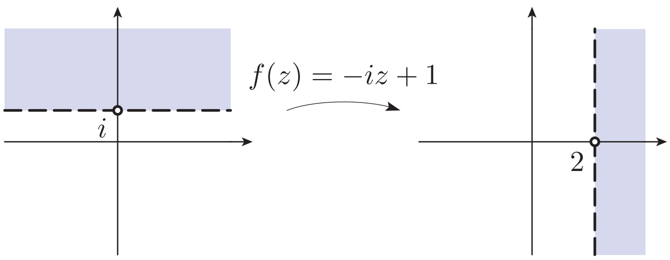
so we define $f(\infty) = \infty$. Hence the extended function is

$$f(z) = \begin{cases} \dfrac{z^2+1}{z}, & z \in \mathbb{C} - \{0\}, \\ \infty, & z \in \{0, \infty\}. \end{cases}$$

Solution to Exercise 1.8

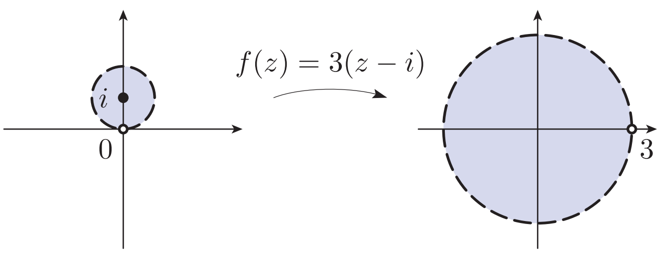
(a) Rotating the half-plane $\{z : \operatorname{Im} z > 1\}$ about 0 through $\pi/2$ clockwise and translating the result one unit to the right gives the half-plane $\{z : \operatorname{Re} z > 2\}$. The linear function that does this is

$$f(z) = e^{-i\pi/2}z + 1 = -iz + 1.$$



(b) Translating the disc $\{z : |z - i| < 1\}$ one unit downwards and then scaling the result by the factor 3 gives the disc $\{z : |z| < 3\}$. The linear function that does this is

$$f(z) = 3(z - i).$$



Solution to Exercise 1.9

(a) Using Theorem 1.2 and Exercise 1.5, we obtain the following classification.

(i) The image of the line $2x - y = 0$ is an (extended) line through the origin (with the origin excluded).

(ii) The image of the circle $x^2 + y^2 = 2$ is a circle not through the origin.

(iii) The image of the circle $x^2 + (y-2)^2 = 4$ is an (extended) line not through the origin.

(iv) The image of the line $2x + y = 1$ is a circle through the origin (with the origin excluded).

(b) **(i)** In $2x - y = 0$, we replace x by $u/(u^2+v^2)$ and y by $-v/(u^2+v^2)$ to obtain

$$\frac{2u}{u^2+v^2} + \frac{v}{u^2+v^2} = 0,$$

that is, $2u + v = 0$.

(ii) In $x^2 + y^2 = 2$, we replace x by $u/(u^2+v^2)$ and y by $-v(u^2+v^2)$ to obtain

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = 2.$$

Then multiplying through by $u^2 + v^2$, we obtain

$$1 = 2(u^2+v^2),$$

that is, $u^2 + v^2 = 1/2$.

Solution to Exercise 1.10

(a) Here f is entire and

$$f(z) = 2z + 6 \to \infty \text{ as } z \to \infty,$$

so we define $f(\infty) = \infty$. Hence the extended function is

$$f(z) = \begin{cases} 2z+6, & z \in \mathbb{C}, \\ \infty, & z = \infty. \end{cases}$$

(b) The function f has a single singularity in $\mathbb{C}$, at 1. Since $f(z) \to \infty$ as $z \to 1$, we define $f(1) = \infty$. Next,

$$f(z) = \frac{1}{z-1} = \frac{1/z}{1 - 1/z}.$$

Therefore

$$f(z) \to \frac{0}{1} = 0 \text{ as } z \to \infty,$$

so we define $f(\infty) = 0$. Hence the extended function is

$$f(z) = \begin{cases} \dfrac{1}{z-1}, & z \in \mathbb{C} - \{1\}, \\ \infty, & z = 1, \\ 0, & z = \infty. \end{cases}$$

Solution to Exercise 2.1

(a) The function $f(z) = 3/z$ is a Möbius transformation with $a = d = 0$, $b = 3$, $c = 1$ and

$$ad - bc = -3 \neq 0.$$

(b) The function $f(z) = (3i + 2z)/z$ is a Möbius transformation with $a = 2$, $b = 3i$, $c = 1$, $d = 0$ and

$$ad - bc = -3i \neq 0.$$

(c) The function $f(z) = 1$ satisfies $f'(z) = 0$ for all complex numbers z, so f is not conformal. Therefore f is not a Möbius transformation, by Theorem 2.1.

(d) The function $f(z) = z + 3/z$ satisfies

$$f'(z) = 1 - 3/z^2.$$

Hence $f'(\sqrt{3}) = 0$, so f is not conformal. Therefore f is not a Möbius transformation, by Theorem 2.1.

(e) The function $f(z) = (1 - i + z)/(2 + 3z)$ is a Möbius transformation with $a = 1$, $b = 1 - i$, $c = 3$, $d = 2$ and

$$ad - bc = 2 - 3(1 - i) = -1 + 3i \neq 0.$$

Solution to Exercise 2.2

(a)

$$f(z) = \begin{cases} \dfrac{2z + i}{-3z + 1}, & z \in \mathbb{C} - \{1/3\}, \\ \infty, & z = 1/3, \\ -2/3, & z = \infty \end{cases}$$

(b)

$$f(z) = \begin{cases} \dfrac{z - i}{3z + 2}, & z \in \mathbb{C} - \{-2/3\}, \\ \infty, & z = -2/3, \\ 1/3, & z = \infty \end{cases}$$

(c)

$$f(z) = \begin{cases} 2z + 1, & z \in \mathbb{C}, \\ \infty, & z = \infty \end{cases}$$

Solution to Exercise 2.3

(a) By Theorem 2.2(a), the Möbius transformation f is a one-to-one function from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$. Also, $f(-2/3) = \infty$ and $f(\infty) = 1/3$. So f maps $\mathbb{C} - \{-2/3\}$ onto $\mathbb{C} - \{1/3\}$. Thus, for each $w \in \mathbb{C} - \{1/3\}$, we have

$$\begin{aligned} w = \frac{z - i}{3z + 2} &\iff 3zw + 2w = z - i \\ &\iff 3zw - z = -2w - i \\ &\iff z = \frac{2w + i}{-3w + 1}. \end{aligned}$$

Hence

$$f^{-1}(w) = \frac{2w + i}{-3w + 1}, \quad \text{for } w \in \mathbb{C} - \{1/3\}.$$

Since $f(\infty) = 1/3$ and $f(-2/3) = \infty$, we have

$$f^{-1}(1/3) = \infty \quad \text{and} \quad f^{-1}(\infty) = -2/3.$$

Hence the inverse function is

$$f^{-1}(w) = \begin{cases} \dfrac{2w + i}{-3w + 1}, & w \in \mathbb{C} - \{1/3\}, \\ \infty, & w = 1/3, \\ -2/3, & w = \infty. \end{cases}$$

Remark: The function f in this exercise is the inverse function of the function f considered in Example 2.1, so you may have expected this answer.

(b) By Theorem 2.2(a), the Möbius transformation f is a one-to-one function from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$. Also, $f(4/3) = \infty$ and $f(\infty) = 1/3$. So f maps $\mathbb{C} - \{4/3\}$ onto $\mathbb{C} - \{1/3\}$. Thus, for each $w \in \mathbb{C} - \{1/3\}$, we have

$$\begin{aligned} w = \frac{z + 2i}{3z - 4} &\iff 3zw - 4w = z + 2i \\ &\iff 3zw - z = 4w + 2i \\ &\iff z = \frac{4w + 2i}{3w - 1}. \end{aligned}$$

Hence

$$f^{-1}(w) = \frac{4w + 2i}{3w - 1}, \quad \text{for } w \in \mathbb{C} - \{1/3\}.$$

Since $f(\infty) = 1/3$ and $f(4/3) = \infty$, we have

$$f^{-1}(1/3) = \infty \quad \text{and} \quad f^{-1}(\infty) = 4/3.$$

Hence the inverse function is

$$f^{-1}(w) = \begin{cases} \dfrac{4w + 2i}{3w - 1}, & w \in \mathbb{C} - \{1/3\}, \\ \infty, & w = 1/3, \\ 4/3, & w = \infty. \end{cases}$$

Solution to Exercise 2.4

Let $f(z) = (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Then, by Theorem 2.2(a), f is a one-to-one function from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$. We deal with the cases $c \neq 0$ and $c = 0$ separately.

Suppose first that $c \neq 0$. Then $f(\infty) = a/c$ and $f(-d/c) = \infty$. So f maps $\mathbb{C} - \{-d/c\}$ onto $\mathbb{C} - \{a/c\}$. Thus, for each $w \in \mathbb{C} - \{a/c\}$, we have

$$\begin{aligned} w = \frac{az+b}{cz+d} &\iff czw + dw = az + b \\ &\iff czw - az = -dw + b \\ &\iff z = \frac{-dw+b}{cw-a} \\ &\iff z = \frac{dw-b}{-cw+a}. \end{aligned}$$

Hence

$$f^{-1}(w) = \frac{dw-b}{-cw+a}, \quad \text{for } w \in \mathbb{C} - \{a/c\}.$$

Since $f(\infty) = a/c$ and $f(-d/c) = \infty$, we have

$$f^{-1}(a/c) = \infty \quad \text{and} \quad f^{-1}(\infty) = -d/c.$$

Hence the inverse function is

$$f^{-1}(w) = \begin{cases} \dfrac{dw-b}{-cw+a}, & w \in \mathbb{C} - \{a/c\}, \\ \infty, & w = a/c, \\ -d/c, & w = \infty, \end{cases}$$

as required. The function f^{-1} is a Möbius transformation because

$$da - (-b)(-c) = ad - bc \neq 0,$$

and because f^{-1} is defined suitably at the pole at a/c and at ∞.

The case $c = 0$ is similar except that this time $f(\infty) = \infty$ and f maps $\mathbb{C}$ onto $\mathbb{C}$, so we obtain

$$f^{-1}(w) = \begin{cases} \dfrac{dw-b}{-cw+a}, & w \in \mathbb{C}, \\ \infty, & w = \infty. \end{cases}$$

Solution to Exercise 2.5

Theorem 2.3 states that we can write down the inverse function of a Möbius transformation $f(z) = (az+b)/(cz+d)$ by interchanging the coefficients a and d, and reversing the signs of the coefficients b and c. By applying this process to the Möbius transformations in Exercise 2.3, we obtain the following inverse functions.

(a) $f^{-1}(w) = \dfrac{2w+i}{-3w+1}$

(b) $f^{-1}(w) = \dfrac{-4w-2i}{-3w+1} = \dfrac{4w+2i}{3w-1}$

Following our convention, these transformations are defined at their poles, and at the point ∞, in the usual way.

Solution to Exercise 2.6

Let z be a complex number that is not a singularity of g and such that $g(z)$ is not a singularity of f. Since $(f \circ g)(z) = f(g(z))$, we have

$$\begin{aligned} (f \circ g)(z) &= f\left(\frac{z+i}{z-i}\right) \\ &= \frac{\left(\dfrac{z+i}{z-i}\right) + 2}{\left(\dfrac{z+i}{z-i}\right) - 2} \\ &= \frac{(z+i) + 2(z-i)}{(z+i) - 2(z-i)} \\ &= \frac{3z-i}{-z+3i}. \end{aligned}$$

Hence $f \circ g$ is the Möbius transformation

$$h(z) = \frac{3z-i}{-z+3i}.$$

Solution to Exercise 2.7

(a) The fixed points of f are ∞ and the solution in $\mathbb{C}$ of $z/2 + 1 = z$, which is 2.

(b) The fixed points of f are the solutions in $\mathbb{C}$ of $1/z = z$, which are 1 and -1.

(c) The fixed points of f are the solutions in $\mathbb{C}$ of

$$\frac{3z+i}{-iz+3} = z.$$

Multiplying through by $-iz+3$ gives

$$3z + i = -iz^2 + 3z.$$

Cancelling $3z$ from both sides, we obtain $i = -iz^2$, which is equivalent to $z^2 = -1$. Hence the fixed points are i and $-i$.

Solution to Exercise 2.8

In each case we use formula (2.2).

(a) $$f(z) = \frac{(z-2)}{(z+2)}\frac{(2i+2)}{(2i-2)} = \frac{z-2}{z+2} \times (-i) = \frac{-iz+2i}{z+2}$$

(b) $$f(z) = \frac{(z-i)}{(z-1)}\frac{(\infty-1)}{(\infty-i)} = \frac{z-i}{z-1}$$

(c) $$f(z) = \frac{(z-\infty)}{(z-1)}\frac{(3i-1)}{(3i-\infty)} = \frac{3i-1}{z-1}$$

(d) $$\begin{aligned} f(z) &= \frac{(z-(1+i))}{(z-\infty)}\frac{(0-\infty)}{(0-(1+i))} \\ &= \frac{z-(1+i)}{-(1+i)} \\ &= -\tfrac{1}{2}(1-i)z+1 \end{aligned}$$

Solution to Exercise 2.9

We find the required transformation by using the Implicit Formula for Möbius Transformations, which in this case is

$$\frac{(z-2)}{(2i+2)}\frac{(z+2)}{(2i-2)} = \frac{(w-i)}{(\infty-1)}\frac{(w-1)}{(\infty-i)} = \frac{w-i}{w-1}.$$

By evaluating the constant term and cross-multiplying, we obtain

$$(z-2)(w-1)(-i) = (w-i)(z+2),$$

that is,

$$-izw + iz + 2iw - 2i = wz + 2w - iz - 2i.$$

On collecting the w terms on the left, we obtain

$$(-1-i)zw + (2i-2)w = -2iz,$$

so

$$w = \frac{-2iz}{(-1-i)z+(2i-2)} = \frac{2iz}{(1+i)z+2(1-i)}.$$

The required Möbius transformation is therefore

$$f(z) = \frac{2iz}{(1+i)z+2(1-i)}.$$

If we wish, we can write this in a simpler form by multiplying the numerator and denominator by $1-i$ to give

$$f(z) = \frac{(2+2i)z}{2z-4i} = \frac{(1+i)z}{z-2i}.$$

Alternatively, we could use the results of parts (a) and (b) of Exercise 2.8 to find the required Möbius transformation, because

$$f = k^{-1} \circ h,$$

where

$$h(z) = \frac{-iz+2i}{z+2} \quad \text{and} \quad k(z) = \frac{z-i}{z-1}.$$

Solution to Exercise 2.10

(a) The function $f(z) = (z-i)/(iz+1)$ is not a Möbius transformation because

$$f(z) = \frac{z-i}{iz+1} = \frac{z-i}{i(z-i)} = -i.$$

It is a constant function.

The function $g(z) = (z-1)/(z+2i)$ is a Möbius transformation with $a=1$, $b=-1$, $c=1$, $d=2i$ and $ad-bc = 2i+1 \neq 0$.

The function $h(z) = z/(2z-i)$ is a Möbius transformation with $a=1$, $b=0$, $c=2$, $d=-i$ and $ad-bc = -i \neq 0$.

(b) By Theorem 2.3, the inverse functions of g and h are

$$g^{-1}(w) = \frac{2iw+1}{-w+1} \quad \text{and} \quad h^{-1}(w) = \frac{-iw}{-2w+1}.$$

Solution to Exercise 2.11

Let z be a complex number that is not a singularity of g and such that $g(z)$ is not a singularity of f. Since $(f\circ g)(z) = f(g(z))$, we have

$$\begin{aligned} (f\circ g)(z) &= f\left(\frac{z+1}{z-1}\right) \\ &= \frac{\left(\frac{z+1}{z-1}\right)-2}{\left(\frac{z+1}{z-1}\right)+i} \\ &= \frac{(z+1)-2(z-1)}{(z+1)+i(z-1)} \\ &= \frac{-z+3}{(1+i)z+(1-i)}. \end{aligned}$$

Hence $f\circ g$ is the Möbius transformation

$$h(z) = \frac{-z+3}{(1+i)z+(1-i)}.$$

Solution to Exercise 2.12

In each case we can use the Explicit Formula for Möbius Transformations to find the required transformation f.

(a) $$f(z) = \frac{(z-1)}{(z-\infty)}\frac{(-1-\infty)}{(-1-1)} = \frac{z-1}{-2} = -\frac{1}{2}z + \frac{1}{2}$$

(b) $$f(z) = \frac{(z-1)}{(z+1)}\frac{(0+1)}{(0-1)} = \frac{-z+1}{z+1}$$

(c) $$f(z) = \frac{(z-(1+i))}{(z-0)}\frac{((2-i)-0)}{((2-i)-(1+i))} = \frac{(2-i)z-(3+i)}{(1-2i)z}$$

Solution to Exercise 2.13

We find the required transformation by using the Implicit Formula for Möbius Transformations, which in this case is

$$\frac{(z-2i)}{(z-1)}\frac{((1+2i)-1)}{((1+2i)-2i)} = \frac{(w-1)}{(w-i)}\frac{((1+i)-i)}{((1+i)-1)}.$$

By evaluating the constant terms and cross-multiplying, we obtain

$$(z-2i)(w-i)(2i) = (w-1)(z-1)(-i),$$

that is,

$$2izw + 4w + 2z - 4i = -iwz + iw + iz - i.$$

On collecting the w terms on the left, we obtain

$$3izw + (4-i)w = (-2+i)z + 3i,$$

so

$$w = \frac{(-2+i)z+3i}{3iz+(4-i)}.$$

Hence the required Möbius transformation is

$$f(z) = \frac{(-2+i)z+3i}{3iz+(4-i)}.$$

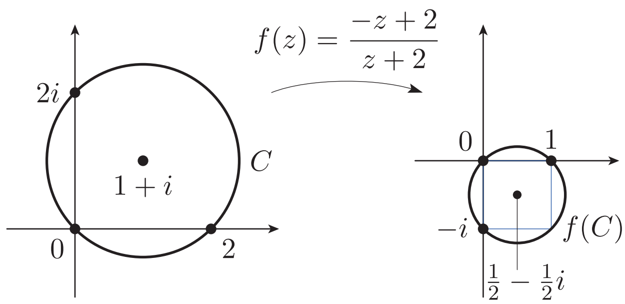
Solution to Exercise 3.1

The circle $C = \{z : |z-(1+i)| = \sqrt{2}\}$ passes through the points 0, 2 and $2i$. Now,

$$f(0) = 1, \quad f(2) = 0 \quad \text{and} \quad f(2i) = \frac{-2i+2}{2i+2} = -i.$$

These points are three of the four vertices of a square centred at $\frac{1}{2} - \frac{1}{2}i$ of side length 1. The image of C is the circle centred at $\frac{1}{2} - \frac{1}{2}i$ of radius $1/\sqrt{2}$, which passes through the four vertices of this square. That is,

$$f(C) = \left\{z : \left|z - \left(\tfrac{1}{2} - \tfrac{1}{2}i\right)\right| = 1/\sqrt{2}\right\}.$$



Solution to Exercise 3.2

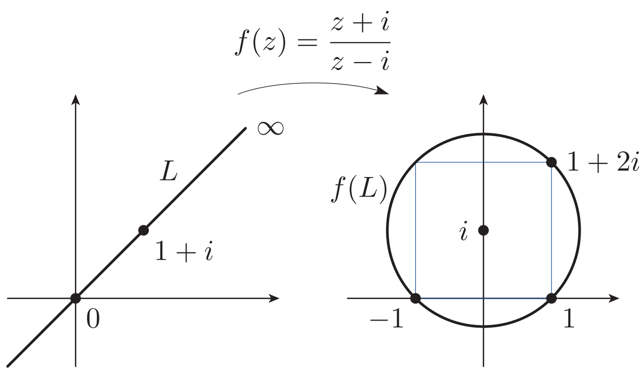
The extended line $L = \{z : \operatorname{Re} z = \operatorname{Im} z\} \cup \{\infty\}$ passes through the points 0, $1+i$ and ∞.

Now,

$$f(0) = -1, \quad f(1+i) = 1+2i \quad \text{and} \quad f(\infty) = 1.$$

These points are three of the four vertices of a square centred at i of side length 2. So the image of L is the circle centred at i of radius $\sqrt{2}$, which passes through the vertices of the square. That is,

$$f(L) = \{z : |z-i| = \sqrt{2}\}.$$



Solution to Exercise 3.3

Here f maps 1 to ∞. Since 1 lies on $C = \{z : |z| = 1\}$, the image of C must be an extended line. In fact, we have

$$f(1) = \infty, \quad f(i) = \frac{i+1}{i-1} = -i \quad \text{and} \quad f(-1) = 0.$$

So $f(C)$ is the extended line through 0 and $-i$. That is, $f(C)$ is the extended imaginary axis

$$\{z : \operatorname{Re} z = 0\} \cup \{\infty\}.$$

Solution to Exercise 3.4

First observe that the image $f(C)$ of $C = \{z : |z| = 1\}$ is an extended line, because $-1 \in C$ and $f(-1) = \infty$. Next, by Theorem 2.3,

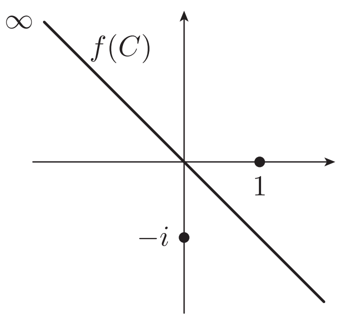
$$f^{-1}(w) = \frac{w+i}{-w+1}.$$

Now, w is a point on the image $f(C)$ if and only if $f^{-1}(w)$ lies on the unit circle C. That is, $w \in f(C)$ if and only if $w = \infty$ or

$$\left|\frac{w+i}{-w+1}\right| = 1.$$

This equation can be rearranged to give $|w+i| = |-w+1|$, which is an equation for the line comprising points that are equidistant from $-i$ and 1. Hence $f(C)$ is the extended line

$$\{w : |w+i| = |-w+1|\} \cup \{\infty\}.$$



This answer is perfectly acceptable, but let us try writing the equation $|w+i| = |-w+1|$ in another form (which will be used later on). By substituting $w = u + iv$ into this equation, we obtain

$$|u + i(v+1)| = |(-u+1) - iv|.$$

Hence

$$u^2 + (v+1)^2 = (-u+1)^2 + v^2.$$

After expanding brackets and cancelling terms we are left with $u = -v$. Therefore, since $u = \operatorname{Re} w$ and $v = \operatorname{Im} w$, we can write the equation $|w+i| = |-w+1|$ as $\operatorname{Re} w = -\operatorname{Im} w$. Hence

$$f(C) = \{w : \operatorname{Re} w = -\operatorname{Im} w\} \cup \{\infty\}.$$

Solution to Exercise 3.5

(a) Since C has equation $|z - (1+i)| = \sqrt{2}$, it follows from the corollary to Theorem 3.1 that $1+i$ and ∞ are inverse points with respect to C. But

$$f(1+i) = i \quad \text{and} \quad f(\infty) = 1.$$

So, by Theorem 3.2, i and 1 are inverse points with respect to $f(C)$.

By Theorem 3.1, $f(C)$ has an equation of the form

$$|w - i| = k|w-1|, \quad \text{for some } k > 0.$$

Since 0 lies on C, it follows that $f(0) = -i$ lies on $f(C)$, so

$$k = \frac{|-i-i|}{|-i-1|} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

An Apollonian form of the equation for $f(C)$, which is a circle, is therefore

$$|w-i| = \sqrt{2}\,|w-1|.$$

(b) Here 1 and ∞ are inverse points with respect to C. So, by Theorem 3.2, $f(1) = 0$ and $f(\infty) = 1$ are inverse points with respect to $f(C)$.

By Theorem 3.1, $f(C)$ has an equation of the form

$$|w| = k|w-1|, \quad \text{for some } k > 0.$$

Since 0 lies on C, it follows that $f(0) = -i$ lies on $f(C)$, so

$$k = \frac{|-i|}{|-i-1|} = \frac{1}{\sqrt{2}}.$$

An Apollonian form of the equation for $f(C)$, which is a circle, is therefore

$$|w| = |w-1|/\sqrt{2} \quad \text{or} \quad |w-1| = \sqrt{2}\,|w|.$$

(c) Here i and ∞ are inverse points with respect to C. So, by Theorem 3.2, $f(i) = \infty$ and $f(\infty) = 1$ are inverse points with respect to $f(C)$.

By Theorem 3.1, $f(C)$ has an equation of the form

$$|w-1| = r, \quad \text{for some } r > 0.$$

Since 0 lies on C, it follows that $f(0) = -i$ lies on $f(C)$, so

$$r = |-i-1| = \sqrt{2}.$$

It follows that $f(C)$ is the circle with equation

$$|w-1| = \sqrt{2}.$$

(d) Here $1+i$ and ∞ are inverse points with respect to C. So, by Theorem 3.2, $f(1+i)=i$ and $f(\infty)=1$ are inverse points with respect to $f(C)$.

By Theorem 3.1, $f(C)$ has an equation of the form

$$|w-i| = k|w-1|, \quad \text{for some } k>0.$$

Since 1 lies on C, it follows that $f(1)=0$ lies on $f(C)$, so

$$k = \frac{|-i|}{|-1|} = 1.$$

An Apollonian form of the equation for $f(C)$, which is an extended line, is therefore

$$|w-i| = |w-1|.$$

Solution to Exercise 3.6

(a) By Theorem 3.3 with $\alpha=i$, $k=\sqrt{2}$ and $\beta=1$, the circle with equation

$$|z-i| = \sqrt{2}\,|z-1|$$

has centre

$$\lambda = \frac{i-2\times 1}{1-2} = 2-i$$

and radius

$$r = \frac{\sqrt{2}\,|i-1|}{|1-2|} = 2.$$

(b) By Theorem 3.3 with $\alpha=1$, $k=\sqrt{2}$ and $\beta=0$, the circle with equation

$$|z-1| = \sqrt{2}\,|z|$$

has centre

$$\lambda = \frac{1-2\times 0}{1-2} = -1$$

and radius

$$r = \frac{\sqrt{2}\,|1-0|}{|1-2|} = \sqrt{2}.$$

Solution to Exercise 3.7

Here C has centre 2 and radius 1, so $\lambda=2$ and $r=1$. So, from Theorem 3.3,

$$\alpha - 2 = \frac{1^2}{\overline{(1+i)-2}} = \frac{1}{\overline{-1+i}} = \frac{-1+i}{2}.$$

Thus $\alpha = \frac{1}{2}(3+i)$, so C has an equation of the form

$$\left|z-\tfrac{1}{2}(3+i)\right| = k|z-(1+i)|.$$

Since 1 lies on C, we find that

$$k = \frac{\left|-\frac{1}{2}-\frac{1}{2}i\right|}{|-i|} = \frac{1}{\sqrt{2}}.$$

Hence an equation for C in Apollonian form is

$$\left|z-\tfrac{1}{2}(3+i)\right| = \frac{1}{\sqrt{2}}|z-(1+i)|.$$

Solution to Exercise 3.8

(a) The reflection of the point $2+3i$ in L is $-2+3i$, so $\alpha=-2+3i$. An equation for L in Apollonian form is therefore

$$|z-(-2+3i)| = |z-(2+3i)|.$$

(b) The reflection of the point $4-2i$ in L is $2-2i$, so $\alpha = 2-2i$. An equation for L in Apollonian form is therefore

$$|z-(2-2i)| = |z-(4-2i)|.$$

Solution to Exercise 3.9

(a) Here f maps 2 to ∞. Since 2 lies on C, the image of C is an extended line. Also,

$$f(-2i)=0 \quad\text{and}\quad f(-2)=\tfrac{1}{2}-\tfrac{1}{2}i.$$

Thus the image $f(C)$ is the extended line through the points 0 and $\frac{1}{2}-\frac{1}{2}i$, namely

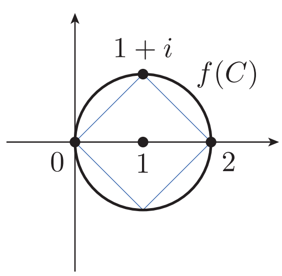
$$\{z : \operatorname{Re} z = -\operatorname{Im} z\} \cup \{\infty\}.$$

(b) Choosing the three points 2, -2 and $2i$ on C, we have

$$f(2)=0, \quad f(-2)=2 \quad\text{and}\quad f(2i)=1+i.$$

These points are three of the four vertices of a square centred at 1 of side length $\sqrt{2}$. Hence $f(C)$ is the circle centred at 1 of radius 1, which passes through the vertices of the square. That is,

$$f(C) = \{z : |z-1| = 1\}.$$



Solution to Exercise 3.10

We use the substitution method for part (a) and the inverse points method for part (b). Both methods can be used for both parts.

(a) The pole $-i$ of f does not lie on the circle $C = \{z : |z + i| = 1\}$, so $f(C)$ must be a circle. By Theorem 2.3,

$$f^{-1}(w) = \frac{iw + i}{-w + 1}.$$

Now, w is a point on the image $f(C)$ if and only if $f^{-1}(w)$ lies on the circle C. That is, $w \in f(C)$ if and only if

$$\left| \frac{iw + i}{-w + 1} + i \right| = 1.$$

Multiplying through by $|-w + 1|$, we obtain

$$|(iw + i) + i(-w + 1)| = |-w + 1|,$$

that is, $|2i| = |w - 1|$. Hence $f(C)$ is the circle

$$|w - 1| = 2.$$

(b) The centre of $C = \{z : |z - i| = 1\}$ is i, so it follows, from the corollary to Theorem 3.1, that i and ∞ are inverse points with respect to C.

Hence, by Theorem 3.2,

$$f(i) = 0 \quad \text{and} \quad f(\infty) = 1$$

are inverse points with respect to $f(C)$. So, by Theorem 3.1, $f(C)$ has an equation of the form

$$|w| = k|w - 1|, \quad \text{for some } k > 0.$$

Since 0 lies on C, $f(0) = -1$ lies on $f(C)$, so

$$k = \frac{|-1|}{|-1 - 1|} = \frac{1}{2}.$$

It follows that $f(C)$ is the circle $|w - 1| = 2|w|$.

Solution to Exercise 3.11

(a) By Theorem 3.3 with $\alpha = i$, $k = 2$ and $\beta = -3i$, the circle with equation

$$|z - i| = 2|z + 3i|$$

has centre

$$\lambda = \frac{i - 4(-3i)}{1 - 4} = -\frac{13}{3}i$$

and radius

$$r = \frac{2|i - (-3i)|}{|1 - 4|} = \frac{8}{3}.$$

(b) By Theorem 3.3 with $\alpha = 0$, $k = 6$ and $\beta = -i$, the circle with equation

$$|z| = 6|z + i|$$

has centre

$$\lambda = \frac{0 - 36(-i)}{1 - 36} = -\frac{36}{35}i$$

and radius

$$r = \frac{6|0 - (-i)|}{|1 - 36|} = \frac{6}{35}.$$

Solution to Exercise 3.12

(a) (i) The circle C_1 has centre $\lambda = 0$ and radius $r = 2$. Thus, using the equation

$$(\alpha - \lambda)\overline{(\beta - \lambda)} = r^2$$

from Theorem 3.3 with $\beta = 1 + i$, we see that

$$\alpha - 0 = \frac{2^2}{\overline{(1 + i) - 0}} = \frac{4(1 + i)}{2},$$

so $\alpha = 2(1 + i)$.

(ii) The circle C_2 has centre $\lambda = i$ and radius $r = \frac{1}{2}$. Thus, from Theorem 3.3 with $\beta = 1 + i$, we see that

$$\alpha - i = \frac{\left(\frac{1}{2}\right)^2}{\overline{(1 + i) - i}} = \frac{1}{4},$$

so $\alpha = \frac{1}{4} + i$.

(b) (i) An Apollonian form of the equation for the circle C_1 is

$$|z - 2(1 + i)| = k|z - (1 + i)|,$$

for some $k > 0$. Since 2 lies on C_1,

$$k = \frac{|2 - 2(1 + i)|}{|2 - (1 + i)|} = \frac{|-2i|}{|1 - i|} = \sqrt{2},$$

so C_1 has equation

$$|z - 2(1 + i)| = \sqrt{2}\,|z - (1 + i)|.$$

(ii) An Apollonian form of the equation for the circle C_2 is

$$\left|z - \left(\tfrac{1}{4} + i\right)\right| = k|z - (1 + i)|,$$

for some $k > 0$. Since $\frac{1}{2}i + i$ lies on C_2,

$$k = \frac{\left|\left(\frac{1}{2} + i\right) - \left(\frac{1}{4} + i\right)\right|}{\left|\left(\frac{1}{2} + i\right) - (1 + i)\right|} = \frac{\left|\frac{1}{4}\right|}{\left|-\frac{1}{2}\right|} = \frac{1}{2},$$

so C_2 has equation

$$\left|z - \left(\tfrac{1}{4} + i\right)\right| = \tfrac{1}{2}|z - (1 + i)|.$$

Solution to Exercise 4.1

(a) $\{z : |z| = 2\}$

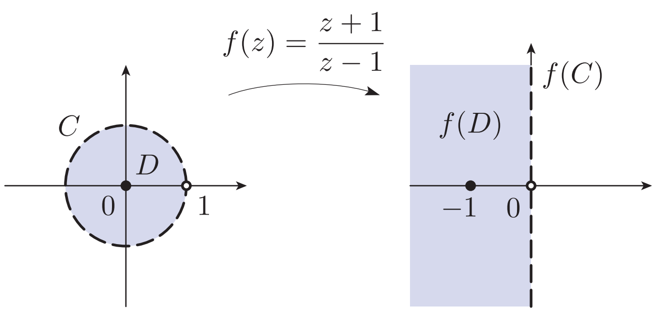
(b) $\{z : \operatorname{Re} z = 1\} \cup \{\infty\}$

(c) $\{z : |z| = 3\} \cup \{\infty\}$

(d) $\{z : |z| = 3\}$

Solution to Exercise 4.2

The boundary in $\widehat{\mathbb{C}}$ of D is the unit circle $C = \{z : |z| = 1\}$. We proved in Exercise 3.3 that $f(C)$ is the extended imaginary axis. It follows from Theorem 4.1 that $f(D)$ is a generalised open disc with boundary in $\widehat{\mathbb{C}}$ the extended imaginary axis. Hence $f(D)$ is either the left or the right half-plane. Since $0 \in D$, and $f(0) = -1$, we see that $f(D)$ is the left half-plane.



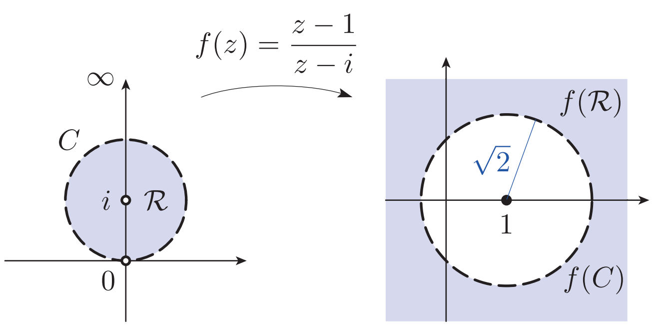
Solution to Exercise 4.3

The boundary in $\widehat{\mathbb{C}}$ of D is the circle $C = \{z : |z - i| = 1\}$. We proved in Exercise 3.5(c) that $f(C)$ is the circle $\{w : |w - 1| = \sqrt{2}\}$. Now, $i \in D$ and $f(i) = \infty$. Hence, by Theorem 4.1,

$$f(D) = \{w : |w - 1| > \sqrt{2}\} \cup \{\infty\}.$$

Since $\mathcal{R} = D - \{i\}$, it follows that

$$f(\mathcal{R}) = f(D) - \{f(i)\} = \{w : |w - 1| > \sqrt{2}\}.$$

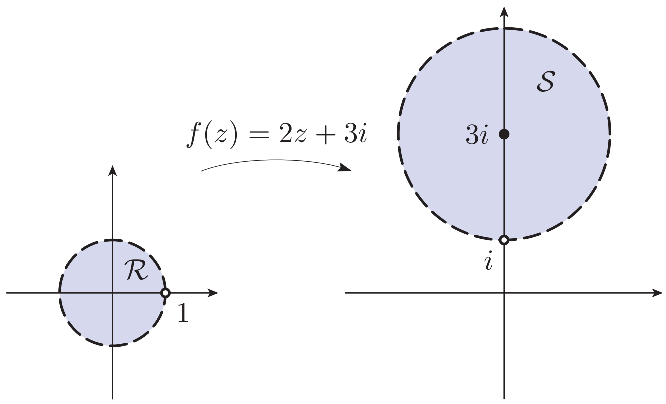


Solution to Exercise 4.4

We can apply a linear function to map $\mathcal{R}$ onto $\mathcal{S}$, as follows. First scale $\mathcal{R}$ by two units, and then translate upwards by 3 units. These two transformations are given by the functions

$$z \longmapsto 2z \quad \text{and} \quad z \longmapsto z + 3i.$$

Hence $f(z) = 2z + 3i$ is a linear function that maps $\mathcal{R}$ onto $\mathcal{S}$, and since f is a linear function, it is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.



Solution to Exercise 4.5

The boundary of $\mathcal{R}$ is the circle

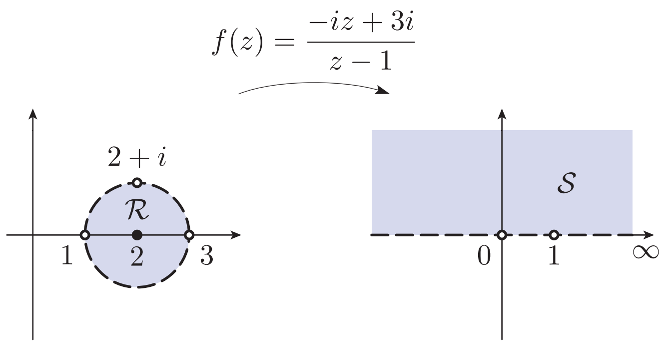
$$C = \{z : |z - 2| = 1\},$$

and the boundary in $\widehat{\mathbb{C}}$ of $\mathcal{S}$ is the extended real line. We will construct a Möbius transformation f that sends the points 3, $2 + i$ and 1, which lie on C, in order to 0, 1 and ∞, which lie on the extended real line.

By the Explicit Formula for Möbius Transformations,

$$\begin{aligned} f(z) &= \frac{(z-3)}{(z-1)}\frac{((2+i)-1)}{((2+i)-3)} \\ &= \left(\frac{z-3}{z-1}\right)\left(\frac{1+i}{-1+i}\right) \\ &= \frac{-iz+3i}{z-1}. \end{aligned}$$

Now, the Möbius transformation f maps C onto the extended real line, and it maps the centre 2 of the disc $\mathcal{R}$ to $f(2) = i$, which lies in $\mathcal{S}$. Both $\mathcal{R}$ and $\mathcal{S}$ are generalised open discs, so it follows from Theorem 4.1 that f maps $\mathcal{R}$ onto $\mathcal{S}$. Since f is a Möbius transformation, it is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.



Solution to Exercise 4.6

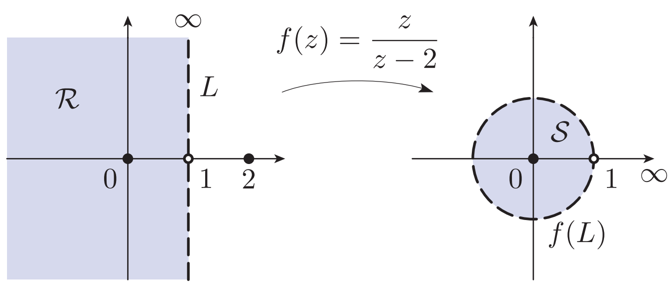
The points 0 and 2 are inverse points with respect to the extended line L that is the boundary of $\mathcal{R}$, and $\infty \in L$. Also, 0 and ∞ are inverse points with respect to the unit circle, and the point 1 lies on the unit circle. So let us choose a Möbius transformation f that satisfies

$$f(0) = 0, \quad f(2) = \infty \quad \text{and} \quad f(\infty) = 1.$$

Using the Explicit Formula for Möbius Transformations, we obtain

$$f(z) = \frac{z}{z-2}.$$

By preservation of inverse points under f, 0 and ∞ are inverse points with respect to $f(L)$, so $f(L)$ is a circle centred at 0. Since $1 \in f(L)$, we see that $f(L)$ is the unit circle. Observe also that $0 \in \mathcal{R}$ and $f(0) = 0$ lies in $\mathcal{S}$. Therefore we can apply Theorem 4.1 to see that the Möbius transformation f is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.



Solution to Exercise 4.7

We can use the square function

$$f(z) = z^2.$$

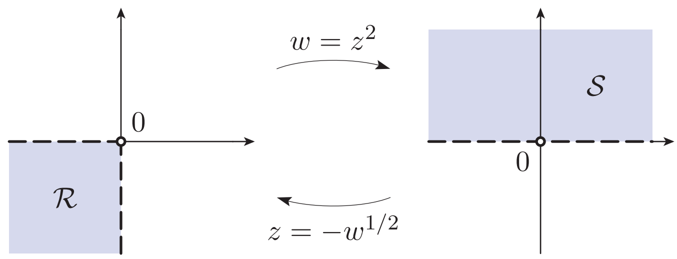
This function squares the modulus of each complex number and doubles the argument. Therefore it is a one-to-one analytic function from

$$\mathcal{R} = \{z : -\pi < \operatorname{Arg} z < -\pi/2\}$$

onto the set of points w with arguments between -2π and $-\pi$. By adding 2π to each of these arguments (to get them in the range of the principal argument), we see that f is a one-to-one conformal mapping from $\mathcal{R}$ onto the upper half-plane

$$\mathcal{S} = \{w : 0 < \operatorname{Arg} w < \pi\}.$$

Next observe that the principal square root function $z = w^{1/2}$ maps $\mathcal{S} = \{w : 0 < \operatorname{Arg} w < \pi\}$ onto the upper-right quadrant $\{z : 0 < \operatorname{Arg} z < \pi/2\}$ (it halves the principal argument of each non-zero complex number). Therefore the (not principal) square root function $z = -w^{1/2}$ maps $\mathcal{S}$ onto the lower-left quadrant $\mathcal{R}$, so this is the inverse function of f.



Solution to Exercise 4.8

Both regions are lunes of angle $\pi/2$ (for each lune, the two generalised circles that bound the lune intersect at right angles), so we can apply the strategy for mapping lunes to find a Möbius transformation f that maps $\mathcal{R}$ onto $\mathcal{S}$.

First choose f to map the vertices of $\mathcal{R}$ to those of $\mathcal{S}$, say $f(1) = 0$ and $f(i) = \infty$. Now, the region $\mathcal{R}$ lies to the left of the upper boundary arc of $\mathcal{R}$ traversed from 1 to i. Also, the region $\mathcal{S}$ lies to the left of the positive real axis traversed from 0 to ∞. We select a point on the upper arc of $\mathcal{R}$ to map to a point on the positive real axis, say $f(e^{i\pi/4}) = 1$.

In summary, we have

$$f(1) = 0, \quad f(e^{i\pi/4}) = 1 \quad \text{and} \quad f(i) = \infty.$$

Using the Explicit Formula for Möbius Transformations, we see that

$$f(z) = \frac{(z-1)}{(z-i)}\frac{(e^{i\pi/4}-i)}{(e^{i\pi/4}-1)}.$$

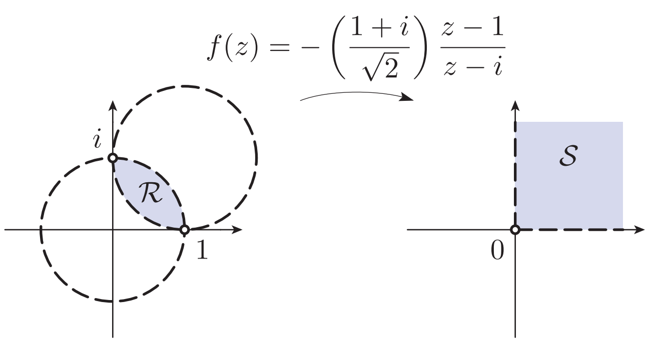
Now,

$$\frac{e^{i\pi/4}-i}{e^{i\pi/4}-1} = \frac{e^{i\pi/4}(1-e^{i\pi/4})}{e^{i\pi/4}-1} = -e^{i\pi/4},$$

and $-e^{i\pi/4} = -(1+i)/\sqrt{2}$, so

$$f(z) = -\left(\frac{1+i}{\sqrt{2}}\right)\frac{z-1}{z-i}.$$

Since f is one-to-one and conformal throughout $\widehat{\mathbb{C}}$, we see that f is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.



Remark: You may have chosen points other than $e^{i\pi/4}$ and 1 when applying the strategy for mapping lunes, to give a different, but equally valid, answer. Also, you may not have obtained the constant $-(1+i)/\sqrt{2}$ (or a similar constant) using the same method presented here.

Notice that the transformation $z \longmapsto kz$, where $k > 0$, is a one-to-one conformal mapping from $\mathcal{S}$ onto $\mathcal{S}$. It follows that we can multiply the formula for $f(z)$ by any positive number k to give another one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

Solution to Exercise 4.9

The cut plane is

$$\mathcal{R} = \{z : |\operatorname{Arg} z| < \pi\}.$$

By applying the principal square root function, $z_1 = \sqrt{z}$, we can halve the angle at the vertex to obtain the right half-plane

$$\mathcal{R}_1 = \{z : |\operatorname{Arg} z| < \pi/2\}.$$

Next, using Table 4.1, the function

$$w = \frac{z_1 - 1}{z_1 + 1}$$

is a one-to-one conformal mapping from $\mathcal{R}_1$ onto the open unit disc $\mathcal{S}$.

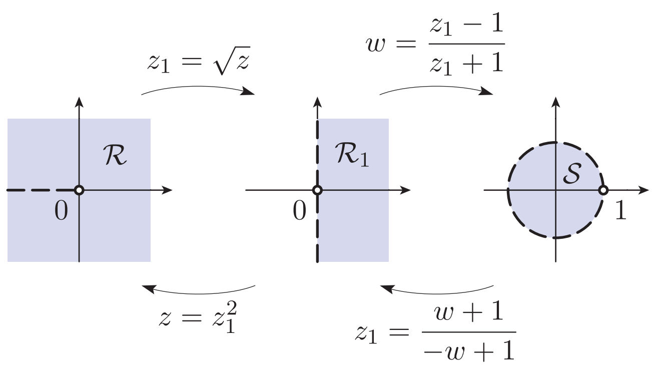
Since both these functions are one-to-one and conformal, the composition of the two functions

$$f(z) = \frac{\sqrt{z}-1}{\sqrt{z}+1}$$

is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$.

For the inverse function we observe that f is one-to-one on $\mathcal{R}$ (because its constituent functions are one-to-one), so f has an inverse function that can be obtained by composing the inverse of the square root function with the inverse of the Möbius transformation. This gives

$$f^{-1}(w) = \left(\frac{w+1}{-w+1}\right)^2.$$



Solution to Exercise 4.10

(a) $\{z : |z - i| = 2\}$

(b) $\{z : \operatorname{Im} z = -1\} \cup \{\infty\}$

(c) $\{z : |z + i| = 1\}$

(d) $\{z : |z + i| = 1\} \cup \{\infty\}$

Solution to Exercise 4.11

(a) The boundary in $\widehat{\mathbb{C}}$ of the disc

$$D = \{z : |z-1| < 2\}$$

is the circle

$$C = \{z : |z-1| = 2\}.$$

The points 3, $1+2i$ and -1 on C are such that

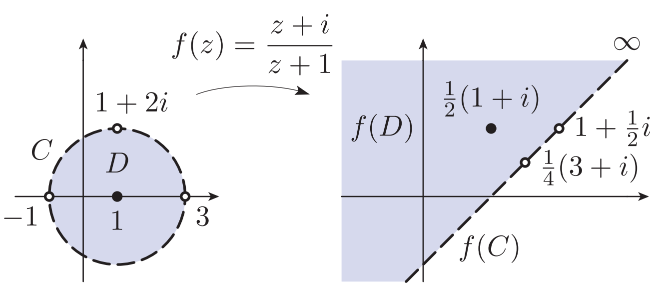
$$f(3) = \frac{3+i}{4} = \tfrac{1}{4}(3+i),$$
$$f(1+2i) = \frac{1+3i}{2+2i} = 1 + \tfrac{1}{2}i,$$
$$f(-1) = \infty.$$

Thus the image of C under f is the extended line that passes through the points $\frac{1}{4}(3+i)$ and $1+\frac{1}{2}i$. That is,

$$f(C) = \left\{z : \operatorname{Re} z - \operatorname{Im} z = \tfrac{1}{2}\right\} \cup \{\infty\}.$$

Since $1 \in D$, and $f(1) = \frac{1}{2}(1+i)$, which lies to the left of $f(C)$ in the complex plane, we see from Theorem 4.1 that $f(D)$ is the generalised open disc

$$f(D) = \left\{z : \operatorname{Re} z - \operatorname{Im} z < \tfrac{1}{2}\right\}.$$



(b) The boundary in $\widehat{\mathbb{C}}$ of $D = \{z : |z+i| < 1\}$ is the circle $C = \{z : |z+i| = 1\}$. We proved in Exercise 3.10(a) that $f(C)$ is the circle $\{w : |w-1| = 2\}$. Now, $-i \in D$ and $f(-i) = \infty$. Hence, by Theorem 4.1, $f(D)$ is the generalised open disc

$$f(D) = \{w : |w-1| > 2\} \cup \{\infty\}.$$

Since $\mathcal{R} = D - \{-i\}$, it follows that

$$f(\mathcal{R}) = f(D) - \{f(-i)\} = \{w : |w-1| > 2\}.$$

Solution to Exercise 4.12

The boundary in $\widehat{\mathbb{C}}$ of the open half-plane

$$H = \{z : \operatorname{Re} z + \operatorname{Im} z < 1\}$$

is the extended line

$$L = \{z : \operatorname{Re} z + \operatorname{Im} z = 1\} \cup \{\infty\}.$$

Observe that 0 and $1+i$ are inverse points with respect to L. Since $0 \in H$, we can ensure (using Theorem 4.1) that H is mapped onto an open disc centred at 0 by mapping 0 to 0 and $1+i$ to ∞. To ensure that H maps onto the open disc with radius 2 (and centre 0), we map ∞ (which is on the boundary in $\widehat{\mathbb{C}}$ of H) to 2. The required Möbius transformation f that does this is

$$f(z) = \frac{2z}{z-(1+i)}.$$

Solution to Exercise 4.13

First we use the lune strategy to map the right-angled lune $\mathcal{R}$ onto the open sector

$$\mathcal{R}_1 = \{z_1 : |\operatorname{Arg} z_1| < \pi/4\}.$$

We choose this sector (rather than, say, the first quadrant) because we can then apply the square function $w = z_1^2$ to map $\mathcal{R}_1$ onto the right half-plane $\mathcal{S}$.

For the lune strategy, we send the vertices i to 0 and $-i$ to ∞. Now, the region $\mathcal{R}$ lies to the left of the left-hand boundary arc of $\mathcal{R}$, traversed from i to $-i$, and the region $\mathcal{R}_1$ lies to the left of the boundary ray with equation $\operatorname{Arg} z_1 = -\pi/4$, traversed from 0 to ∞. We choose a point $1-\sqrt{2}$ on the left-hand boundary arc of $\mathcal{R}$ and send it to the point $e^{-i\pi/4}$ on the boundary ray of $\mathcal{R}_1$. Using the Implicit Formula for Möbius Transformations with $\alpha = i$, $\beta = 1-\sqrt{2}$, $\gamma = -i$ and $\alpha' = 0$, $\beta' = e^{-i\pi/4}$, $\gamma' = \infty$, we obtain

$$\frac{(z-i)}{(z+i)}\frac{(1-\sqrt{2}+i)}{(1-\sqrt{2}-i)} = \frac{(z_1-0)}{(z_1-\infty)}\frac{(e^{-i\pi/4}-\infty)}{(e^{-i\pi/4}-0)} = \frac{z_1}{e^{-i\pi/4}}.$$

(We chose α, β, γ in this particular order to make the calculation of z_1 easier. Choosing α, β, γ in a different order will give a different, but equally valid, Möbius transformation.)

Since $1+i=\sqrt{2}e^{i\pi/4}$ and $1-i=\sqrt{2}e^{-i\pi/4}$, we see that

$$e^{-i\pi/4}\left(\frac{1-\sqrt{2}+i}{1-\sqrt{2}-i}\right)=e^{-i\pi/4}\left(\frac{\sqrt{2}e^{i\pi/4}-\sqrt{2}}{\sqrt{2}e^{-i\pi/4}-\sqrt{2}}\right)$$
$$=\frac{\sqrt{2}-\sqrt{2}e^{-i\pi/4}}{\sqrt{2}e^{-i\pi/4}-\sqrt{2}}=-1.$$

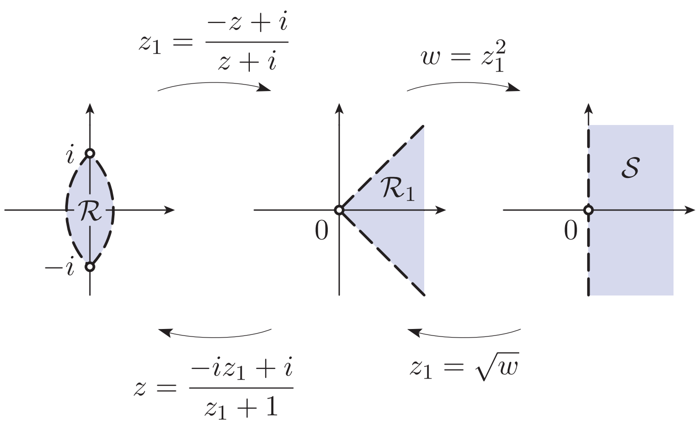
Hence

$$z_1=-\frac{z-i}{z+i}=\frac{-z+i}{z+i}$$

is a Möbius transformation that maps $\mathcal{R}$ onto $\mathcal{R}_1$. Composing this with the square function, we obtain the mapping

$$f(z)=\left(\frac{-z+i}{z+i}\right)^2,$$

which is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$, since it is composed of one-to-one conformal mappings from $\mathcal{R}$ onto $\mathcal{R}_1$ and from $\mathcal{R}_1$ onto $\mathcal{S}$.



Since f is one-to-one, it has an inverse function that we can obtain by composing the inverse functions of the constituent mappings. The inverse function of the function $w=z_1^2$ is $z_1=\sqrt{w}$, and the inverse function of the Möbius transformation

$$z_1=\frac{-z+i}{z+i}$$

is

$$z=\frac{-iz_1+i}{z_1+1}.$$

Hence

$$f^{-1}(w)=\frac{-i\sqrt{w}+i}{\sqrt{w}+1}.$$

References

Bottazzini, U. and Gray, J. (2013) *Hidden Harmony – Geometric Fantasies: The Rise of Complex Function Theory*, Sources and Studies in the History of Mathematics and Physical Sciences, New York, Springer.

Gray, J. (2015) *The Real and the Complex: A History of Analysis in the 19th Century*, Springer Undergraduate Mathematics Series, Cham, Switzerland, Springer.

Painlevé, P. (1967) *Analyse des Travaux Scientifiques Jusqu'en 1900*, Paris, Librairie Scientifique et Technique Albert Blanchard.

Remmert, R. (1998) *Classical Topics in Complex Function Theory* (trans. from German by L. Kay), Graduate Texts in Mathematics, vol. 172, New York, Springer.

Acknowledgements

Grateful acknowledgement is made to the following sources.

Unit C1

Bernhard Riemann (end of Section 5): Universal History Archive / Contributor / Getty Images

Lars Valerian Ahlfors (end of Section 5): Mathematisches Institut Oberwolfach. This file is licensed under the Creative Commons license: https://creativecommons.org/licenses/by-sa/2.0/de/deed.en

Maryam Mirzakhani (end of Section 5): David Neff/Contributor/Getty Images

Unit C2

Hermann Amandus Schwarz (end of Section 4): Paul Fearn/Alamy

Unit C3

August Ferdinand Möbius (end of Subsection 2.1): Chronicle/Alamy

Hermann Minkowski (end of Section 2): Pictorial Press Ltd/Alamy

Every effort has been made to contact copyright holders. If any have been inadvertently overlooked the publishers will be pleased to make the necessary arrangements at the first opportunity.

Index